TOPICS ON THE GEOMETRY OF HOMOGENEOUS SPACES

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Abstract. This is a survey paper about a selection of results in complex algebraic geometry that appeared in the recent and less recent litterature, and in which rational homogeneous spaces play a prominent rôle. This selection is largely arbitrary and mainly reflects the interests of the author.

Rational homogeneous varieties are very special projective varieties, which appear in a variety of circumstances as exhibiting extremal behavior. In the quite recent years, a series of very interesting examples of pairs (sometimes called Fourier-Mukai partners) of derived equivalent, but not isomorphic, and even non birationally equivalent manifolds have been discovered by several authors, starting from the special geometry of certain homogeneous spaces. We will not discuss derived categories and will not describe these derived equivalences: this would require more sophisticated tools and much ampler discussions. Neither will we say much about Homological Projective Duality, which can be considered as the unifying thread of all these apparently disparate examples. Our much more modest goal will be to describe their geometry, starting from the ambient homogeneous spaces.

In order to do so, we will have to explain how one can approach homogeneous spaces just playing with Dynkin diagram, without knowing much about Lie theory. In particular we will explain how to describe the VMRT (variety of minimal rational tangents) of a generalized Grassmannian. This will show us how to compute the index of these varieties, remind us of their importance in the classification problem of Fano manifolds, in rigidity questions, and also, will explain their close relationships with prehomogeneous vector spaces.

We will then consider vector bundles on homogeneous spaces, and use them to construct interesting birational transformations, including important types of flops: the Atiyah and Mukai flops, their stratified versions, also the Abuaf-Segal and Abuaf-Ueda flops; all these beautiful transformations are easily described in terms of homogenenous spaces. And introducing sections of the bundles involved, we will quickly arrive at several nice examples of Fourier-Mukai partners.

We will also explain how the problem of finding crepant resolutions of orbit closures in prehomogeneous spaces is related to the construction of certain manifolds with trivial canonical class. This gives a unified perspective over classical constructions by Reid, Beauville-Donagi and Debarre-Voisin of abelian and hyperKähler varieties, naturally embedded into homogeneous spaces. The paper will close on a recent construction, made in a similar spirit, of a generalized Kummer fourfold from an alternating three-form in nine variables.

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1. Basics

1.1. **Rational homogeneous spaces.** A classical theorem of Borel and Remmert [Ak95] asserts that a projective complex manifold which admits a transitive action of its automorphism group is a direct product of an abelian variety by a rational homogeneous space. The latter can be described as a quotient G/P, where G is a semi-simple algebraic group and P a *parabolic subgroup.* Moreover it can always be decomposed into a product

$$G/P \simeq G_1/P_1 \times \cdots \times G_\ell/P_\ell$$

of rational homogeneous spaces of simple algebraic groups $G_1, \ldots, G_\ell$.

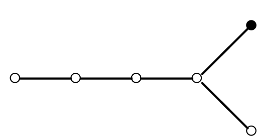
So we will suppose in the sequel, unless otherwise stated, that G is simple. Moreover the list of homogeneous spaces under G will only depend on its Lie algebra $\mathfrak{g}$, which is clasically encoded in a Dynkin diagram Δ. In fact the G-equivalence classes of parabolic subgroups are in bijective correspondence with the finite subsets of nodes of Δ. As a conclusion, a projective rational homogeneous space with simple automorphism group is determined by a marked Dynkin diagram.

The two extremes cases correspond to the *complete flag manifolds* (all nodes marked), and the *generalized Grassmannians* (only one node marked). Generalized Grassmannians are equivariantly embedded inside the projectivizations of the fundamental representations, and from this perspective they are exactly their geometric counterparts. (For a quick introduction to the Lie theoretic background, see e.g. [Ma13]). In type A_n, we get the usual Grassmannians:

$$\simeq \quad G/P = G(3,8) \subset \mathbb{P}(\wedge^3\mathbb{C}^8)$$

In types B_n, D_n (resp. C_n) there is an invariant quadratic (resp. symplectic) form preserved by the group G, and the generalized Grassmannians G/P_k are $OG(k,m)$, for $m = 2n+1$ or $2n$ (resp. $IG(k,2n)$), the subvarieties of the usual Grassmannians parametrizing *isotropic subspaces.*

This has to be taken with a grain of salt for $k = n$ or $n-1$ in type D_n (and also for $k = n$ in type B_n): the variety of isotropic spaces $OG(n,2n) \subset \mathbb{P}(\wedge^n\mathbb{C}^{2n})$ has two connected components; moreover, the restriction L of the Plücker line bundle to a component is divisible by two, and the line bundle M such that $L = 2M$ embeds this component into the projectivization of a *half- spin representation* Δ_{2n}. As embedded varieties, these two components are in fact undistinguishable. We will denote them by $\mathbb{S}_{2n} \subset \mathbb{P}(\Delta_{2n})$ and call them the *spinor varieties.*



$$\simeq \quad G/P = \mathbb{S}_{12} \subset \mathbb{P}(\Delta_{12})$$

1.2. **Some reasons to care.** Being homogeneous, homogeneous spaces could look boring! But we are far from knowing everything about them. For example:

(1) Their Chow rings are not completely understood, except for usual Grassmannians and a few other varieties: *Schubert calculus* has been an intense field of research since the 19th century, involving geometers, representation theoretists, and combinatorists! Modern versions include K-theory, equivariant cohomology, quantum cohomology, equivariant quantum K-theory...

Interesting *quantum to classical* principles are known in some cases, which for instance allow to deduce the quantum cohomology of Grassmannians from the usual intersection theory on two-step flag varieties.

(2) Derived categories are fully described only for special cases (Grassmannians, quadrics, isotropic Grassmannians of lines, a few other sporadic cases), although important progress have been made for classical Grassmannians.

(3) Characterizations of homogeneous spaces are important but not known (except in small dimension). This is one of the potential interests of rational homogeneous spaces for the algebraic geometer: their behavior is often extremal in some sense. Here are some important conjectures:

Campana-Peternell conjecture (1991). *Let X be a smooth complex Fano variety with nef tangent bundle. Then X is homogeneous.*

See [MOSWW15] for a survey. Nefness is a weak version of global generation: varieties with globally generated tangent bundles are certainly homogeneous. Note that varieties with ample tangent bundles are projective spaces: this was conjectured by Hartshorne and Frankel, and proved by Mori.

A smooth codimension one distribution on a variety Y is defined as a corank one sub-bundle H of the tangent bundle TY. Let $L = TY/H$ denote the quotient line bundle. The Lie bracket on TY induces a linear map $\wedge^2 H \longrightarrow L$. This gives what is called a *contact structure* when this skew-symmetric form is non degenerate (this implies that the dimension $d = 2n+1$ of Y is odd and that the canonical line bundle $\omega_Y = -(n+1)L$, see [KPSW00]) .

Lebrun-Salamon conjecture (1994). *Let Y be a smooth complex Fano variety admitting a contact structure. Then Y is homogeneous.*

Hartshorne's conjecture (1974) in complex projective geometry is that if $Z \subset \mathbb{P}^N$ is smooth, non linearly degenerate, of dimension $n > \frac{2N-2}{3}$, then Z must be a complete intersection. In particular it must be linearly normal, meaning that it is not a projection from $\mathbb{P}^{N+1}$. Although Hartshorne's conjecture is still widely open, this last statement was indeed shown by Zak, who also proved [Zak93]:

Zak's theorem II (1981). *Let Z be a smooth, non linearly degenerate, non linearly normal subvariety of $\mathbb{P}^N$, of dimension $n = \frac{2N-2}{3}$. Then Z is homogeneous.*

1.3. **Homogeneous spaces and Fano manifolds.**

The importance of homogeneous spaces in the theory of Fano varieties comes from the following

Fact. *Generalized Grassmannians are Fano manifolds of high index.*

Definition 1.1. *Let X be a Fano manifold of dimension n and Picard number one, so that $Pic(X) = \mathbb{Z}L$ for some ample line bundle L, and $K_X = -i_X L$. Then i_X is the index and the coindex $c_X := n + 1 - i_X$.*

Theorem 1.2. (1) *If $c_X = 0$, then $X \simeq \mathbb{P}^n$ is a projective space (Mori).*
(2) *If $c_X = 1$, then $X \simeq \mathbb{Q}^n$ is a quadric (Miyaoka).*

Here is a list of generalized Grassmannians of coindex 2 (del Pezzo manifolds) and 3 (Mukai varieties).

		dimension	index
del Pezzo	$G(2,5)$	6	5
Mukai	G_2/P_2	5	3
	$IG(3,6)$	6	4
	$G(2,6)$	8	6
	$\mathbb{S}_{10}$	10	8

Observation. Suppose L is very ample and embeds X in $\mathbb{P}V$. Then a smooth hyperplane section $Y = X \cap H$ has the same coindex $c_Y = c_X$.

As a consequence, a generalized Grassmannian contains, as linear sections, a large family of Fano submanifolds of dimension down to c_X. Moreover, these families are always *locally complete*, in the sense that every small complex deformation of any member of the family is of the same type. In particular, Mukai varieties allow to construct locally complete families of prime Fano threefolds of index 1.

Theorem 1.3 (Fano-Iskhovskih's classification [IP99])**.** *Any prime Fano threefold of index one is either:*

(1) *a complete intersection in a weighted projective space,*
(2) *a quadric hypersurface in a del Pezzo homogeneous space,*
(3) *a linear section of a homogeneous Mukai variety,*
(4) *a trisymplectic Grassmannian.*

For the last case, one gets Fano threefolds in $G(3,7)$, parametrizing spaces that are isotropic with respect to three alternating two-forms. A unified description of the last three cases is by zero loci of sections of homogeneous bundles on homogeneous spaces; this stresses the importance of understanding those bundles.

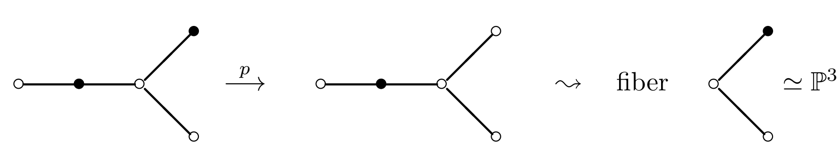
1.4. **Correspondences and lines.** There are natural projections from (generalized) flag manifolds to Grassmannians. In particular, given two generalized Grassmannians G/P and G/Q of the same complex Lie group G, we can connect them through a flag manifold $G/(P \cap Q)$ (we may suppse that $P \cap Q$ is still parabolic):

$$\begin{array}{ccc} & G/(P\cap Q) & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ G/P & & G/Q \end{array}$$

The fiber of p is $P/(P \cap Q)$, in particular it is homogeneous. If its automorphism group is the semisimple Lie group L, we can write $P/(P \cap Q) = L/R$ and this generalized flag manifold is determined as follows.

- Start with the Dynkin diagram Δ of G, with the two nodes δ_P and δ_Q defining G/P and G/Q.
- Suppress the node δ_P and the edges attached to it.
- Keep the connected component of the remaining diagram that contains δ_Q.

The resulting marked Dynkin diagram is that of the fiber $P/(P \cap Q) = L/R$.

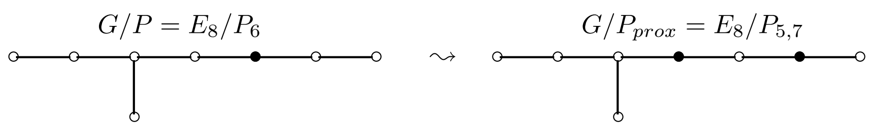


This rule has an obvious extension to the case where G/P and G/Q are not necessarily generalized Grassmannians.

In particular, start with a generalized Grassmannian G/P, defined by the marked Dynkin diagram (Δ, δ_P). Let δ_P^{prox} be the set of vertices in Δ that are connected to δ_P. Let G/P_{prox} be the generalized flag manifold defined by the marked Dynkin diagram $(\Delta, \delta_P^{prox})$. Then the fibers of q are projective lines!

Theorem 1.4. [LM03] *If δ_P is a long node of Δ (in particular, if Δ is simply laced), then G/P_{prox} is the variety of lines on G/P.*

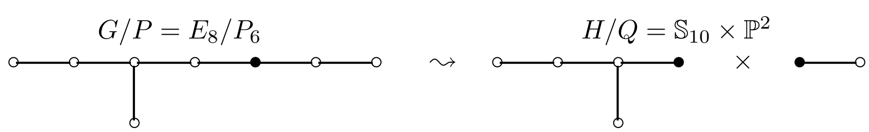
If δ_P is a short node of Δ, then the variety of lines on G/P is irreducible with two G-orbits, and G/P_{prox} is the closed one.



In particular, any generalized Grassmannians parametrizes linear spaces on a generalized Grassmannian defined by an extremal node of a Dynkin diagram.

There is a similar statement for the variety of lines passing through a fixed point x of $G/P \subset \mathbb{P}V$. These lines are parametrized by a subvariety $\Sigma_x \subset \mathbb{P}(T_xX)$, independant of x up to projective equivalence. Moreover Σ_x is stable under the action of the stabilizer $P_x \simeq P$ of x. The semisimple part H of P has Dynkin diagram $\Delta_H = \Delta - \{\delta_P\}$, and we can define a parabolic subgroup $Q \subset H$ by marking in Δ_H the vertices that in Δ were connected to δ_P.

Theorem 1.5. [LM03] *If δ_P is a long node of Δ (in particular, if Δ is simply laced), then $\Sigma_x \simeq H/Q$. If δ_P is a short node of Δ, then Σ_x is made of two H-orbits, and H/Q is the closed one.*



Variety of lines through fixed points in homogeneous spaces are instances of the so-called VMRTs (varieties of minimal rational tangents), which have been extensively studied in the recent years in connexion with rigidity problems (see e.g. [Hw19] for an introduction).

From deformation theory, one knows that the variety of lines passing through a general point of a Fano manifold X has dimension $i_X - 2$. With the previous notations, this gives a nice way to compute the index of a generalized Grassmannian:

$$i_{G/P} = \dim(L/R) + 2.$$

Example 1. Let us consider the rank two (connected) Dynkin diagrams.

(1) $\Delta = A_2$, $G = PGL_3$. Then the two generalized Grassmannians are projective planes, and each of them parametrizes the lines in the other one.
(2) $\Delta = B_2$, $G = PSO_5$. The two generalized Grassmannians are the quadric $\mathbb{Q}^3$ and $OG(2,5)$. The latter parametrizes the lines in $\mathbb{Q}^3$. But note that

$OG(2,5)$ has dimension 3 and anticanonical twice the restriction of the Plücker line bundle, hence index 4 since this restriction is 2-divisible. So by Mori's theorem $OG(2,5) \simeq \mathbb{P}^3$! Moreover, lines in $\mathbb{P}^3$ are parametrized by $G(2,4) \simeq \mathbb{Q}^4$, and $\mathbb{Q}^3$ is the closed PSO_5-orbit.

(3) $\Delta = C_2$, $G = PSp_4$. The two generalized Grassmannians are $\mathbb{P}^3$ and $IG(2,4)$. The latter is a hyperplane section of $G(2,4) \simeq \mathbb{Q}^4$, hence a copy of $\mathbb{Q}^3$. Of course we recover the previous case.

(4) $\Delta = G_2$, $G = G_2$. The two generalized Grassmannians G_2/P_1 and G_2/P_2 have the same dimension 5, but different indexes 5 and 3. In particular by Miyaoka's theorem $G_2/P_1 \simeq \mathbb{Q}^5$. Moreover G_2/P_2 must be the closed G_2-orbit in the variety of lines in $\mathbb{Q}^5$, which is $OG(2,7)$.

1.5. **Sporadic examples.** There exist a few series of generalized Grassmannians with strikingly similar properties.

Severi varieties. These are the four varieties

$$v_2(\mathbb{P}^2) \subset \mathbb{P}^5, \quad \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \quad G(2,6) \subset \mathbb{P}^{14}, \quad E_6/P_1 \subset \mathbb{P}^{26}.$$

Here v_2 means that we consider the second Veronese embedding. Each of these varieties is the singular locus of a special cubic hypersurface (the secant variety), and the derivatives of this cubic define a quadro-quadric Cremona tranformation. Note also that they are varieties of dimension $2a$ embedded inside $\mathbb{P}^{3a+2}$ for $a = 1,2,4,8$. These are the homogeneous spaces that appear in Zak's Theorem II [Zak93].

Legendrian varieties. These are the four varieties

$$IG(3,6) \subset \mathbb{P}^{13}, \quad G(3,6) \subset \mathbb{P}^{19}, \quad \mathbb{S}_{12} \subset \mathbb{P}^{31}, \quad E_7/P_1 \subset \mathbb{P}^{55}.$$

Each of these varieties is contained the singular locus of a special quartic hypersurface (the tangent variety), and the derivatives of this quartic define a cubo-cubic Cremona tranformation. Moreover they have the remarkable *one apparent double point property*, which means that through a general point of the ambient projective space passes exactly one bisecant. Note also that they are varieties of dimension $3a + 3$ embedded inside $\mathbb{P}^{6a+9}$ for $a = 1,2,4,8$. Finally, their varieties of lines through a given point are nothing else than the Severi varieties [LM07]!

2. Borel-Weil theory and applications

Let G/P be a generalized Grassmannian and suppose that G is simply connected. Let L be the ample generator of the Picard group. Then L is in fact very ample and G-linearizable. In particular $H^0(G/P,L)$ is a G-module and G/P is G-equivariantly embedded inside the dual linear system $|L|^\vee = \mathbb{P}H^0(G/P,L)^\vee$. A typical example is the Plücker embedding of a Grassmannian (see e.g. [FH91] for basics in representation theory).

2.1. **The Borel-Weil theorem.** More generally, on any generalized flag manifold G/Q, any line bundle M is G-linearizable, so $H^0(G/Q,M)$ is a G-module. Moreover M is generated by global sections as soon as it is nef.

Theorem 2.1 (Borel-Weil)**.** *For any nef line bundle M on G/Q, the space of sections $H^0(G/Q,M)$ is an irreducible G-module.*

A line bundle M on G/Q is defined by a weighted version of the marked Dynkin diagram that defines G/Q. Moreover, it is nef/globally generated (resp. ample/very ample) exactly when the weights are non negative (resp. positive), and then the G-module $H^0(G/Q, M)$ is defined by the *same* weighted diagram.

Starting from a projection $p : G/(P \cap Q) \longrightarrow G/P$, by homogeneity the sheaf $E_P = p_*M$ is a G-equivariant vector bundle on G/P. Symmetrically there is a vector bundle $E_Q = q_*M$ on G/Q, and

$$H^0(G/Q, E_Q) = H^0(G/Q, M) = H^0(G/P, E_P).$$

Such identifications allow to play with sections in different homogeneous spaces and describe nice correspondances between their zero loci.

Example 2. Consider the flag manifold $F(2,3,5)$ with its two projections p and q to $G(2,5)$ and $G(3,5)$. The tautological and quotient bundles U_2, Q_2 on $G(2,5)$, U_3, Q_3 on $G(3,5)$, pull-back to vector bundles on $F(2,3,5)$ for which we keep the same notations. The minimal very ample line bundle on $F(2,3,5)$ is $L = \det(U_2)^\vee \otimes \det(Q_3)$. Its push-forwards to the two Grassmannians are $E_2 = \det(U_2)^\vee \otimes \wedge^2 Q_2 \simeq Q_2^*(2)$ on $G(2,5)$ and $E_3 = \wedge^2 U_3^\vee \otimes \det(Q_3) \simeq U_3(2)$ on $G(3,5)$. These are two vector bundles with determinant $\mathcal{O}(5)$. As a consequence, a general section s of L defines two Calabi-Yau threefolds

$$Z_2(s) \subset G(2,5) \qquad \text{and} \qquad Z_3(s) \subset G(3,5).$$

Proposition 2.2. [KR17] *The Calabi-Yau threefolds $Z_2(s)$ and $Z_3(s)$ are derived equivalent, but not birationally equivalent in general.*

Derived equivalent means that their derived categories of coherent sheaves are equivalent as triangulated categories. For two smooth projective varieties X_1 and X_2, this is a very strong property, which implies that they are in fact isomorphic as soon as one of them has an ample or anti-ample canonical bundle (Bondal-Orlov). Non isomorphic but derived equivalent varieties with trivial canonical bundle recently attracted considerable attention, and the previous example of Fourier-Mukai partners is among the simplest.

Example 3. A three-form $\omega \in \wedge^3(\mathbb{C}^n)^\vee$ defines a section of $\wedge^3 T^\vee$ on $G(k,n)$ for any $k \geq 3$, and sections of $Q^\vee(1)$ on $G(2,n)$ and $\wedge^2 Q^\vee(1)$ on $\mathbb{P}^{n-1}$. The latter leads to Pfaffian loci in $\mathbb{P}^{n-1}$. The previous one gives *congruences of lines* in $G(2,n)$: the zero locus of a general section of $Q^\vee(1)$ is a $(n-2)$-dimensional prime (for $n \neq 6$) Fano manifold of index 3. For $n = 6$, one actually gets $\mathbb{P}^2 \times \mathbb{P}^2$.

For $n = 7$, this congruence of lines is isomorphic with G_2/P_2. Indeed, the stabilizer in SL_7 of a general $\omega \in \wedge^3(\mathbb{C}^7)^\vee$ is isomorphic to G_2. This stabilizer acts on the associated congruence, and since there is no non trivial closed G_2-orbit of dimension smaller than five, this congruence has to be a G_2-Grassmannian. But it is not $G_2/P_1 = \mathbb{Q}^5$, whose index is 5, so it must be G_2/P_2.

For $n = 8$, a general $\omega \in \wedge^3(\mathbb{C}^8)^\vee$ is equivalent to the three-form

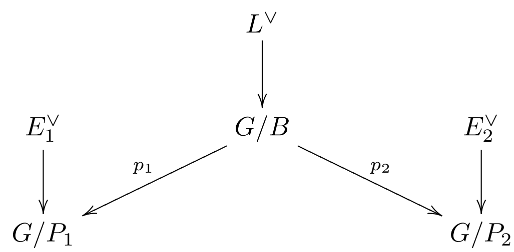
$$\omega_0(x,y,z) = \mathrm{trace}(x[y,z])$$

on $\mathfrak{sl}_3$, which is stabilized by the adjoint action of SL_3. The congruence of lines defined by ω_0 is the variety of *abelian* planes inside $\mathfrak{sl}_3$, a smooth compactification of SL_3/T, for $T \subset SL_3$ a maximal torus: so a quasi-homogeneous, but not homogeneous sixfold.

These constructions intend to illustrate the general

Principle. *Vector bundles on generalized flag manifolds allow to easily construct interesting varieties, notably Fano or Calabi-Yau.*

2.2. **Flops.** Consider a rank two Dynkin diagram Δ, with corresponding group G. The flag varieties of G are the complete flag variety G/B and the two generalized Grassmannians G/P_1 and G/P_2. Let L denote the minimal ample line bundle on G/B. Its push-forwards to G/P_1 and G/P_2 are two vector bundles E_1 and E_2.



Consider a point in the total space of $L^\vee$, that is a pair (x, ϕ) with $x \in G/B$ and ϕ a linear form on the fiber L_x. Let $y = p_1(x)$ and $e \in E_{1,y}$, the corresponding fiber of E_1. Since $E_1 = p_{1*}L$, we can see e as a section of L over $p_1^{-1}(y)$, then evaluate this section at x, and apply ϕ. This defines a morphism $v_1 : Tot(L^\vee) \longrightarrow Tot(E_1^\vee)$. Moreover two points (x, ϕ) and (x', ϕ') have the same image if and only if $p_1(x) = p_1(x')$ and $\phi = \phi' = 0$. This yields:

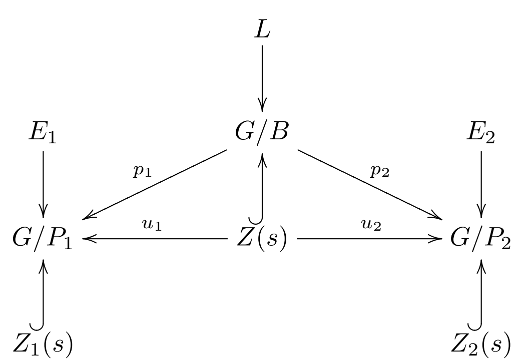
Proposition 2.3. *The morphism $v_i : Tot(L^\vee) \longrightarrow Tot(E_i^\vee)$ is the blowup of G/P_i.*

In particular we get a nice birational map between $Tot(E_1^\vee)$ and $Tot(E_2^\vee)$. In type $A_1 \times A_1$ this is the classical *Atiyah flop* (in dimension three), and in type A_2 the classical *Mukai flop* (dimension four). In type $B_2 = C_2$ this is the *Abuaf-Segal flop* (dimension five) [Se16], and in type G_2 the *Abuaf-Ueda flop* (dimension seven) [Ue19]. An unusual feature of the two latter flops is that they are quite non symmetric, the exceptional loci on both sides being rather different.

Theorem 2.4. *All these flops are derived equivalences.*

This supports a famous conjecture of Bondal and Orlov according to which varieties connected by flops should always be derived equivalent.

The three bundles L, E_1, E_2 have the same space of global sections V. So if we pick a (general) section $s \in V$, we get three zero-loci $Z(s), Z_1(s), Z_2(s)$ and a diagram:



Observe that p_1 and p_2 are $\mathbb{P}^1$-fibrations, such that L is a relative hyperplane bundle. So E_1 and E_2 are rank two bundles. Moreover, s vanishes either at a unique point of a fiber, or on this whole fiber. This implies that u_1 and u_2 are birational morphisms, with exceptional divisors that are $\mathbb{P}^1$-bundles over the codimension two subvarieties $Z_1(s)$ and $Z_2(s)$. More precisely:

Proposition 2.5. *The morphism $u_i : Z(s) \longrightarrow G/P_i$ is the blowup of $Z_i(s)$.*

Note that the canonical bundle of G/B is $-2L$, so that $Z(s)$ is Fano: this is one of the (not so many) examples of blowup of a Fano manifold that remains Fano.

In type A_2, $Z(s)$ is a del Pezzo surface of degree 6, each $Z_i(s)$ consists in three points in a projective plane, and the birationality between the two planes is the classical Cremona transformation. In type C_2, $Z_1(s)$ is a quintic elliptic curve E in $C_2/P_1 = \mathbb{P}^3$, and we obtain one of the examples of the Mori-Mukai classification of Fano threefolds with $b_2 = 2$. The birational map to $C_2/P_2 = \mathbb{Q}^3$ is defined by the linear system of cubics through E. Moreover $Z_2(s)$ is again an elliptic curve, isomorphic to E since both curves can be identified with the intermediate Jacobian of the Fano threefold $Z(s)$. More to come about type G_2.

Remark. In general, consider a projection $p : G/P \longrightarrow G/Q$ and a line bundle L on G/P. The vector bundle $E = p_*L$ is a G-equivariant bundle, defined by a P-module W. Consider a section s of L, vanishing along a smooth hypersurface $Z(s) \subset G/P$. The fiber over x of the projection $p_s : Z(s) \longrightarrow G/Q$ is the zero locus of the restriction s_x of s to the fibers of G/P. Its isomorphism type only depends on the P-orbit of s_x in W. This is one of the many reasons to be interested in representations with finitely many orbits.

2.3. **Consequences for the Grothendieck ring of varieties.** The Grothendieck ring of (complex) varieties is the ring generated by isomorphism classes of algebraic varieties (no scheme structures) modulo the relations

(1) $[X] = [Y] + [X/Y]$ for $Y \subset X$ a closed subvariety,
(2) $[X \times X'] = [X] \times [X']$.

An easy consequence is that if X is a Zariski locally trivial fiber bundle over Z, with fiber F, then $[X] = [Z] \times [F]$. If we denote by $\mathbb{L}$ the class of the affine line, the usual cell decomposition of projective spaces yields

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n = \frac{1 - \mathbb{L}^{n+1}}{1 - \mathbb{L}}.$$

Since the complete flag manifold $\mathbb{F}_n = SL_n/B$ is a composition of projective bundles, one deduces that

$$[\mathbb{F}_n] = \frac{(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^{n+1})}{(1 - \mathbb{L})^n}.$$

Similar formulas exist for any G/P: they admit stratifications by affine spaces (cell decompositions) and the classes $[G/P]$ are therefore polynomials in $\mathbb{L}$, which admit nice factorizations as rational functions.

Let us come back to Proposition 2.5. Since the the blowup u_i gives a $\mathbb{P}^1$-bundle over $Z_i(s)$, and an isomorphim over its complement, we get that

$$[Z(s)] = [G/P_i - Z_i(s)] + [Z_i(s)] \times [\mathbb{P}^1] = [G/P_i] + [Z_i(s)] \times \mathbb{L}.$$

We deduce the identity

$$([Z_1(s)] - [Z_2(s)]) \times \mathbb{L} = [G/P_2] - [G/P_1] = 0.$$

(In fact G/P_1 and G/P_2 have the same Hodge numbers as a projective space: such varieties are called minifolds). This has the unexpected consequence that [IMOU19]:

Theorem 2.6. *$\mathbb{L}$ is a zero divisor in the Grothendieck ring of varieties.*

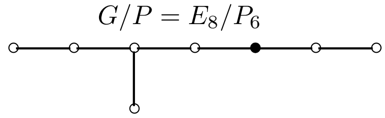
Proof. Let $G = G_2$. One needs to check that in this case, $[Z_1(s)] - [Z_2(s)] \neq 0$. This requires more sophisticated arguments. First, one shows that for s general, $Z_1(s)$ and $Z_2(s)$ are smooth Calabi-Yau threefolds [IMOU19]. Second, their Picard groups are both cyclic, but the minimal ample generators have different degrees. So $Z_1(s)$ and $Z_2(s)$ are not isomorphic.

Therefore they are not birational, because Calabi-Yau's are minimal models, so they would be isomorphic in codimension one, and since the Picard groups are cyclic their minimal generators would match. Therefore they are not stably birational either, because if $Z_1(s)\times\mathbb{P}^m$ was birational to $Z_2(s)\times\mathbb{P}^m$, then their MRC fibrations, or maximal rationally connected fibrations, would also be birational; but since they are not uniruled, $Z_1(s)$ and $Z_2(s)$ are the basis of these fibrations. Finally, a deep result of Larsen and Lunts implies that $[Z_1(s)] - [Z_2(s)]$ does not belong to the ideal generated by $\mathbb{L}$, and in particular it must be non zero. □

The fact that the Grothendieck ring is not a domain was first observed by Poonen (2002). That $\mathbb{L}$ is a zero divisor was first shown by Borisov using the Pfaffian-Grassmannian equivalence [Bo18] (see Proposition 3.4).

2.4. **Representations with finitely many orbits.** Consider a semisimple Lie group G, and an irreducible representation W such that $\mathbb{P}(W)$ contains only finitely many G-orbits. Over $\mathbb{C}$ these representations were classified by Kac [Kac80], who proved that most of them are *parabolic*. Parabolic representations are exactly those representations spanned by the varieties of lines through a given point of a generalized Grassmannian. These representations always admit finitely many orbits. This implies that there are very strong connections between rational homogeneous spaces and prehomogeneous spaces [Ma13].

$G/P = E_8/P_6$



$$\rightsquigarrow \qquad \Delta_{16} \times \mathbb{C}^3 \curvearrowleft Spin_{10} \times GL_3$$

Example 4. For dimensional reasons the spaces of three-forms $\wedge^3(\mathbb{C}^n)^\vee$ cannot have finitely many GL_n-orbits when $n \geq 9$. They do have finitely many GL_n-orbits for $n \leq 8$ because they are parabolic, coming from generalized Grassmannians of type E_n. Orbits were classified long ago.

Note that each orbit is a locally closed subvariety, whose boundary is a union of smaller dimensional orbits. So there is a natural Hasse diagram encoding, for each orbit closure, which other ones are the components of its boundary.

For $n = 6$ the orbit closures are particularly simple to describe. The Hasse diagram is just a line

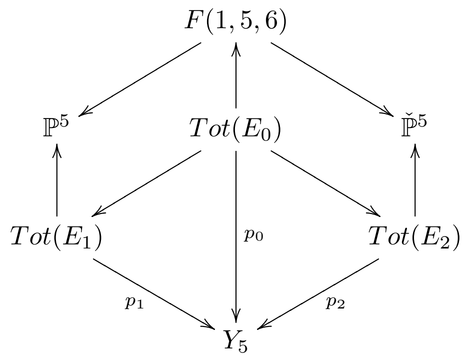
$$Y_0 \longrightarrow Y_1 \longrightarrow Y_5 \longrightarrow Y_{10} \longrightarrow Y_{20},$$

where the index is the codimension:

Y_0	*ambient space*
Y_1	*tangent quartic hypersurface*
Y_5	*singular locus of the quartic*
Y_{10}	*cone over the Grassmannian $G(3,6)$*
Y_{20}	*origin*

Alternatively, Y_5 can be described as parametrizing those three-forms θ that factor as $\omega \wedge \ell$ for some vector ℓ and some two-form ω. In general the line $L = \langle \ell \rangle$ is uniquely defined by θ. Moreover ω (which is defined modulo L) is a two-form of maximal rank 4, supported on a uniquely defined hyperplane H. Then θ belongs to $\wedge^2 H \wedge L$. We conclude that Y_5 is birational to the total space of the rank six vector bundle $E_0 = \wedge^2 U_5 \wedge U_1$ over the flag variety $F(1,5,6)$.

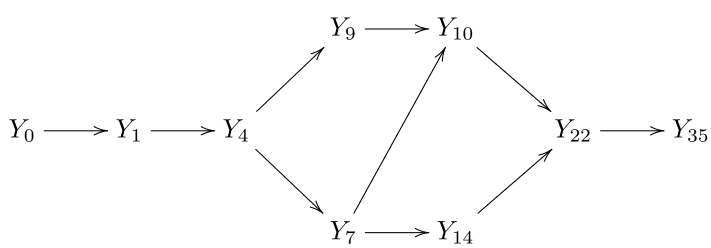
But we can also forget U_1 and U_5 and we get a diagram



where E_1 is the vector bundle $\wedge^2\mathbb{C}^6 \wedge U_1$ and E_2 is the vector bundle $\wedge^3 U_5$, both of rank ten. It is a nice exercize to describe this flop. One checks in particular that

Proposition 2.7. *p_0, p_1, p_2 are resolutions of singularities. p_0 is divisorial, while p_1 and p_2 are crepant resolutions, with exceptional loci of codimension three.*

For $n = 7$ the situation is slightly more complicated. Here is the Hasse diagram, where as previously the index stands for the codimension.



For $n = 8$ there are 22 orbits, partially described in [BMT19].

2.5. **Applications to linear sections.** A nice case is when the generalized Grassmannian G/P is defined by a node that splits the Dynkin diagram Δ into the union of two diagrams, one of which is of type A_{k-1}, with $k \geq 1$ ($k = 1$ occurs when the node is at an end of the diagram). This means that the lines through some fixed point in G/P are parametrized by $\mathbb{P}^{k-1} \times S$ for some homogeneous space S (of some smaller Lie group H).

If the minimal homogeneous embedding of S is inside $\mathbb{P}(U)$ for some irreducible H-module U, the minimal homogeneous embedding of $\mathbb{P}^{k-1} \times S$ is inside $\mathbb{P}(\mathbb{C}^k \otimes U)$. Moreover, by Kac's results mentionned above, the action of $GL_k \times H$ on $\mathbb{C}^k \otimes U$ admits finitely many orbits. This implies that the Grassmannian $G(k, U)$ also admits finitely many H-orbits, and therefore:

Proposition 2.8. *There exist only finitely many isomorphism types of codimension k linear sections of $S \subset \mathbb{P}(U)$.*

Of course the open orbit yields smooth linear sections, but it may happen that other orbits also give smooth sections, although degenerate in some sense. A very interesting case is that of the spinor variety $\mathbb{S}_{10}$ [Ku18b, BFM18]. Starting from the Dynkin diagram of type E_8, we see that linear sections of codimension up to three have finitely many isomorphism types. One can check that there exists only one type of smooth sections of codimension 1, but two different types of codimension 2 and four types of codimension 3.

This gives examples of non locally rigid Fano varieties of high index. Conversely, the local deformations of Fano linear sections of generalized Grassmannians are always of the same type, and the generic Fano linear sections are locally rigid if and only if their marked Dynkin diagrams can be extended to a Dynkin diagram by an arm of length $k-1$ [BFM18]. Actually this has to be taken with a grain of salt: the generic codimension two linear section of the Grassmannian $G(2, 2m+1)$ is also locally rigid, while according to the previous rule, it would come from the diagram E_{2m+2}, which is not Dynkin if $m > 3$! But these are the only exceptions.

3. Some remarkable varieties with trivial canonical bundle

3.1. **More crepant resolutions.** Resolutions of singularities of orbit closures by total spaces of homogeneous vector bundles on generalized flag varieties are instances of *Kempf collapsings*; we have already met three of them in Example 4.

The general construction is extremely simple. Let E be a homogeneous bundle on G/P, whose dual bundle is globally generated, and let $V^\vee := H^0(G/P, E^\vee)$. Then $Tot(E)$ embeds inside $X \times V$, and the projection to V is the Kempf collapsing of E:

$$\begin{array}{ccc} Tot(E) & \longrightarrow & Y \subset V \\ \downarrow & & \\ X & & \end{array}$$

These collapsings have nice properties, in particular when the bundle E is irreducible [Kem76, Wey03]. They are always proper, so the image Y is a closed subset of V, usually singular, and the Kempf collapsing often provides a nice resolutions of singularities.

Sometimes, these resolutions are even *crepant.* One can show that this happens exactly when $\det(E) = \omega_X$ [BFMT17]. Here are a few examples:

Determinantal loci. Let V, W be vector spaces of dimensions v, w, and consider inside $Hom(V, W)$ the subvariety D_k of morphisms of rank at most k. This means that the image is contained in (generically equal to) a k-dimensional subspace U of

W. So D_k is the image of a Kempf collapsing

$$\begin{array}{ccc} Tot(Hom(V,U_k)) & \longrightarrow & D_k \subset Hom(V,W) \\ \downarrow & & \\ G(k,W) & & \end{array}$$

where U_k is the rank k tautological bundle on the Grassmannian. This construction is essential for constructing minimal resolutions of determinantal loci [Wey03]. This Kempf collapsing is crepant if and only if V and W have the same dimension, that is, for square matrices, independently of k.

Cubic polynomials. Inside the spaces C_k of cubic polynomials in k variables, consider the set C_k^2 of those that can be written as polynomials in only two variables. This is the image of the Kempf collapsing

$$\begin{array}{ccc} Tot(S^3U_2) & \xrightarrow{p_1} & C_k^2 \subset C_k \\ \downarrow & & \\ G(2,k) & & \end{array}$$

Since $\det(S^3U_2) = \det(U_2)^6$, the morphism p_1 is crepant if and only if $k = 6$.

Skew-symmetric three-forms. Inside the spaces F_k of skew-symmetric three-forms in k variables, consider the set F_k^6 of those that can be written as three-forms in only six variables. This is the image of the Kempf collapsing

$$\begin{array}{ccc} Tot(\wedge^3U_6) & \xrightarrow{p_2} & F_k^6 \subset F_k \\ \downarrow & & \\ G(6,k) & & \end{array}$$

Since $\det(\wedge^3U_6) = \det(U_6)^{10}$, the morphism p_2 is crepant if and only if $k = 10$.

Pencils of quadrics. Inside the spaces P_k of pencils of quadrics in k variables, consider the set P_k^ℓ of those that can be written in terms of ℓ variables only. This is the image of the Kempf collapsing

$$\begin{array}{ccc} Tot(\mathbb{C}^2 \otimes Sym^2U_\ell) & \xrightarrow{p_3} & P_k^\ell \subset P_k \\ \downarrow & & \\ G(\ell,k) & & \end{array}$$

Since $\det(\mathbb{C}^2 \otimes Sym^2U_\ell) = \det(U_\ell)^{2\ell+2}$, the morphism p_3 is crepant if and only if $k = 2\ell + 2$.

3.2. **Beauville-Donagi type constructions.** Crepancy has the following nice consequence. Suppose that the crepancy condition is fulfilled for the vector bundle E on G/P, that is, $\omega_X = \det(E)$. Then by adjunction, a general section s of $E^\vee$ will vanish on a smooth subvariety $Z(s) \subset G/P$ with trivial canonical bundle.

Let us revisit the previous three examples from this perspective.

Cubic polynomials. A global section s of $S^3U_2^\vee$ on $G(2,6)$ is a cubic polynomial in six variables, and defines a cubic fourfold $X \subset \mathbb{P}^5$. The variety $Z(s) \subset G(2,6)$ is the variety of lines $F(X)$ on this cubic fourfold. This is another fourfold, with trivial canonical bundle, and Beauville and Donagi proved that it is hyperKähler ([BD85], more on this below).

Skew-symmetric three-forms. A global section s of $\wedge^3U_6^\vee$ on $G(6,10)$ is an alternating three-form in ten variables, which also defines a hypersurface X in $G(3,10)$. Then $Z(s) \subset G(6,10)$ can be interpreted as the parameter space for the copies of $G(3,6)$ contained in X. This is again a fourfold, with trivial canonical bundle, and Debarre and Voisin proved that it is hyperKähler [DV10]. Note the strong analogies with the previous case.

Remark. These two examples might give the impression that it should be easy to construct hyperKähler manifolds as zero loci of sections of homogeneous bundles on (generalized) Grassmannians. This impression is completely false. For instance, Benedetti proved that if a hyperKähler fourfold can be described as the zero locus of a general section of a semisimple homogeneous bundle on a Grassmannian, then it has to be one of the two previous examples [Be18].

Pencils of quadrics. Consider a pencil $P \simeq \mathbb{P}^1$ of quadrics in k variables. In general it contains exactly k singular quadrics, which have corank one. For $k = 2\ell+2$ even, the family of ℓ-dimensional projective spaces that are contained in some quadric of the pencil defines a hyperelliptic curve $C \xrightarrow{\eta} P$ of genus ℓ. Moreover, the family of $(\ell-1)$-dimensional projective spaces that are contained in the base locus of the pencil is a ℓ-dimensional smooth manifold A with trivial canonical bundle. Reid proved that A is an abelian variety, isomorphic with the Jacobian variety of C [Re72, Theorem 4.8].

3.3. **The Springer resolution and Richardson nilpotent orbits.** A fundamental homogeneous vector bundle on G/P is the cotangent bundle $\Omega_{G/P}$. Its dual, the tangent bundle, is generated by global sections. Moreover, it is a general fact that the space of global sections of the tangent bundle is the Lie algebra of the automorphism group. Beware that the automorphism group of G/P can very-well be bigger than G (we have seen that for $\mathbb{Q}^5 = G_2/P_1 = B_3/P_1$), but except for a few well understood exceptions, we have $H^0(G/P, T_{G/P}) = \mathfrak{g}$.

In any case, G/P being G-homogeneous, there is always an injective map from $\mathfrak{g}$ to $H^0(G/P, T_{G/P})$, whose image is a linear system of sections that generate the tangent bundle at every point. In particular the Kempf collapsing of $\Omega_{G/P}$ to $\mathfrak{g}$ is well-defined. Note moreover that the fiber of $T_{G/P}$ over the base point is $\mathfrak{g}/\mathfrak{p}$, whose dual identifies with $\mathfrak{p}^\perp \subset \mathfrak{g}^\vee \simeq \mathfrak{g}$ (where the duality is given by the Cartan-Killing form). One can check that $\mathfrak{p}^\perp$ is the nilpotent radical of $\mathfrak{p}$, in particular it is made of nilpotent elements of $\mathfrak{g}$.

One denotes by $\mathcal{N}$ the *nilpotent cone* in $\mathfrak{g}$. A fundamental result in Lie theory is that the adjoint action of G on $\mathcal{N}$ has finitely many orbits. This implies that for each parabolic P, the image of the Kempf collapsing of the cotangent bundle of

G/P is the closure of a uniquely defined nilpotent orbit $\mathcal{O}_P$:

$$\begin{array}{ccc} Tot(\Omega_{G/P}) & \xrightarrow{\pi_P} & \bar{\mathcal{O}}_P \subset \mathcal{N} \subset \mathfrak{g} \\ \downarrow & & \\ G/P & & \end{array}$$

Note that the crepancy condition is automatically fulfilled. In fact both sides of the collapsing have natural symplectic structures. These are preserved by π_P, but beware that this collapsing is not neccessarily birational, although it is in most cases. The most important one is the case where $P = B$, which yields the so-called *Springer resolution* [CG97]

$$\begin{array}{ccc} Tot(\Omega_{G/B}) & \xrightarrow{\pi_B} & \mathcal{N} \subset \mathfrak{g} \\ \downarrow & & \\ G/B & & \end{array}$$

This is a fundamental example of a symplectic resolution of singularities.

3.4. **Stratified Mukai flops.** In some situations the same orbit closure has several distinct symplectic resolutions. This happens in particular when the Dynkin diagram of G has a non trivial symmetry, that is in type A, D or E_6. In type A, the cotangent bundles of the Grassmannians $G(k, n)$ and $G(n-k, n)$ resolve the same nilpotent orbit closure, so there is a birational morphism between them which is called a *stratified Mukai flop* (of type A).

$$\begin{array}{ccccc} & & F(k, n-k, n) & & \\ & \swarrow & \uparrow & \searrow & \\ G(k,n) & & Tot(E_0) & & G(n-k,n) \\ \uparrow & \swarrow & & \searrow & \uparrow \\ Tot(\Omega_{G(k,n)}) & & & & Tot(\Omega_{G(n-k,n)}) \\ & \searrow & & \swarrow & \\ & & \bar{\mathcal{O}}_{k,n} & & \end{array}$$

where $\bar{\mathcal{O}}_{k,n} \subset \mathfrak{sl}_n$ is the set of nilpotent matrices of square zero and rank at most k. In particular $k = 1$ gives a *Mukai flop*. The other cases yield the so-called stratified Mukai flops of types D and E. In particular the cotangent spaces of the two spinor varieties of type D_n resolve the same nilpotent orbit [Na08, Fu07].

3.5. **Projective duality.** Consider an embedded projective variety $X \subset \mathbb{P}(V)$ and its projective dual $X^* \subset \mathbb{P}(V^\vee)$. By definition X^* parametrizes the tangent hyperplanes to X. If the latter is smooth, this implies that the affine cone over X^*

is the image of the Kempf collapsing

$$\begin{array}{ccc} Tot((\hat{T}X)^{\perp}) & \xrightarrow{\quad\pi\quad} & \hat{X}^* \subset V^{\vee} \\ \downarrow & & \\ X & & \end{array}$$

where we denoted by $\hat{T}X$ the affine tangent bundle of X, which is a sub-bundle of the trivial bundle with fiber V. If π is generically finite, then $X^* \subset \mathbb{P}(V^{\vee})$ is a hypersurface: this is the general expectation. Moreover projective duality is an involution, and this implies that the general fibers of π are projective spaces. In particular, if X^* is indeed a hypersurface, then the projectivization of π is a resolution of singularities.

Now suppose that V and its dual are G-modules with finitely many orbits. Since projective duality is compatible with the G-action, it must define a bijection between the G-orbit closures in $\mathbb{P}(V)$ and those in $\mathbb{P}(V^{\vee})$ (but not necessarily compatible with inclusions).

3.6. **Cubic-K3 and Pfaffian-Grassmannian dualities.** To be even more specific, consider $G = GL_n$ acting on the space $V = \wedge^2(\mathbb{C}^n)^{\vee}$ of alternating bilinear forms. The orbits are defined by the rank, which can be any even integer between 0 and n. We denote by Pf_r the orbit closure consisting of forms of rank at most r. These orbits can also be seen inside $V^{\vee}$. One has

$$(Pf_r)^* = Pf_{n-r-\epsilon},$$

with $\epsilon = 0$ if n is even, $\epsilon = 1$ if n is odd. For n even, Pf_{n-2} is the Pfaffian hypersurface, of degree $n/2$. But if n is odd, there is no invariant hypersurface, and the complement Pf_{n-3} of the open orbit has codimension three.

$n = 5$. The PGL_5-orbits in $\mathbb{P}(\wedge^2\mathbb{C}^5)$ are $G(2,5)$ and its complement. In particular $G(2,5)$ is projectively self-dual. Note that its index is 5, so its intersection

$$X_g = G(2,5) \cap gG(2,5)$$

with a translate by a general $g \in PGL(\wedge^2\mathbb{C}^5)$ is a smooth Calabi-Yau threefold. This is also true for the intersection of their projective duals,

$$Y_g = G(2,5)^* \cap (gG(2,5))^*.$$

These two Calabi-Yau's are obviously deformation equivalent, but [BCP17, OR18]:

Theorem 3.1. *For $g \in PGL(\wedge^2\mathbb{C}^5)$ general:*

1. *X_g and Y_g are derived equivalent.*
2. *X_g and Y_g are not birationally equivalent.*
3. *In the Grothendieck ring of varieties, $([X_g] - [Y_g]) \times \mathbb{L}^4 = 0$.*

Note moreover that Example 2 is a limit case, obtained when the two Grassmannians collapse one to the other along some normal direction (indeed the normal bundle to $G(2,5) \subset \mathbb{P}^9$ is isomorphic with $Q^{\vee}(2)$). Interestingly, for this case one obtains in the Grothendieck ring the stronger relation $([Z_1(s)] - [Z_2(s)]) \times \mathbb{L}^2 = 0$.

Similar phenomena can be observed if one replaces the Grassmannian $G(2,5)$ by the spinor variety $\mathbb{S}_{10}$ [Ma18].

$n = 6$. The proper PGL_6-orbit closures in $\mathbb{P}(\wedge^2\mathbb{C}^6)$ are the Grassmannian $G(2,6)$ and the Pfaffian hypersurface. The projective dual of $G(2,6)$ is the Pfaffian hypersurface Pf_4 inside $\mathbb{P}(\wedge^2\mathbb{C}^6)^\vee \simeq \mathbb{P}(\wedge^4\mathbb{C}^6)$, which is singular exactly along $G(4,6)$. Since the codimension of the latter is 6, the intersection of Pf_4 with a general five dimensional projective space $L \subset \mathbb{P}(\wedge^4\mathbb{C}^6)$ is a smooth cubic fourfold X_L. Moreover $L^\perp \subset \mathbb{P}(\wedge^2\mathbb{C}^6)$ has codimension 6, and it follows that its intersection with $G(2,6)$ is a smooth K3 surface S_L. Beauville and Donagi [BD85] showed that:

Proposition 3.2. *The Fano variety of lines $F(X_L) \simeq Hilb^2(S_L)$.*

In particular $F(X_L)$ is hyperKähler, and since this property is preserved by deformation, the variety of lines of any cubic fourfold is also hyperKähler, as soon as it is smooth.

Proof. Take two points in S_L, defining two planes P_1 and P_2 in $\mathbb{C}^6$. In general these planes are transverse and we can consider $Q = P_1 \oplus P_2$. The alternating form that vanish on Q (which implies they are degenerate) span a $\mathbb{P}^5$ in $\mathbb{P}(\wedge^2\mathbb{C}^6)^\vee$, which is automatically orthogonal to P_1 and P_2. So the intersection with L is defined by only four conditions (rather than 6), and the intersection is in general a line in X_L. This defines a rational map $a : Hilb^2(S_L) \longrightarrow F(X_L)$, which is in fact regular.

Conversely, take a line ℓ in X_L. In particular this is a line parametrizing alternating forms of rank four, and if the rank drops to two X_L must be singular, which we exclude. By [MM05, Proposition 2], we have:

Lemma 3.3. *Up to the action GL_6, a line of alternating forms of constant rank four in six variables can be put in one of the two possible normal forms*

$$\langle e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_5 + e_2 \wedge e_6\rangle, \quad \textit{or} \quad \langle e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_6\rangle.$$

In particular, there always exists a unique four plane Q (the orthogonal to e_1 and e_2) on which all the forms in the line ℓ vanish. Note that $\mathbb{P}(\wedge^2 Q)^\perp \cap L$ is a linear space in X_L that contains ℓ, so it must coincide with ℓ unless X_L contains a plane. By dimension count, this implies that $\mathbb{P}(\wedge^2 Q) \cap L^\perp$ is also a line, which has to cut the quadric $G(2,Q) \subset \mathbb{P}(\wedge^2 Q)$ along two points (more precisely along a length two subscheme). These two points belong to S_L, so this defines a map $b : F(X_L) \longrightarrow Hilb^2(S_L)$, which is inverse to a. ☐

$n = 7$. The proper PGL_7-orbit closures in $\mathbb{P}(\wedge^2\mathbb{C}^7)$ are the Grassmannian $G(2,7)$ and the Pfaffian variety of degenerate forms, of codimension three. The projective dual of $G(2,7)$ is the Pfaffian variety Pf_4 inside $\mathbb{P}(\wedge^2\mathbb{C}^7)^\vee \simeq \mathbb{P}(\wedge^5\mathbb{C}^7)$, which is singular exactly along $G(5,7)$. The codimension of the latter is 10, so the intersection of Pf_4 with a general six dimensional projective space $L \subset \mathbb{P}(\wedge^5\mathbb{C}^7)$ is a smooth threefold X_L. Moreover $L^\perp \subset \mathbb{P}(\wedge^2\mathbb{C}^7)$ has codimension 7, so its intersection with $G(2,7)$ is a smooth Calabi-Yau threefold Y_L.

Theorem 3.4. *Suppose that X_L and Y_L are smooth. Then:*
1. *X_L and Y_L are derived equivalent Calabi-Yau threefolds.*
2. *X_L and Y_L are not birationally equivalent.*
3. *In the Grothendieck ring of varieties, $([X_L] - [Y_L]) \times \mathbb{L}^6 = 0$.*

This is the famous *Pfaffian-Grassmannian equivalence* [BC09]. The identity in the Grothendieck ring is due to Martin.

Proof. The fact that X_L has trivial canonical bundle can be proved by observing that the minimal resolution of the Pfaffian locus $Pf_4 \subset \mathbb{P} := \mathbb{P}(\wedge^5\mathbb{C}^7)$ is the Buchsbaum-Eisenbud complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-7) \stackrel{\wedge^3\omega}{\longrightarrow} \mathcal{O}_{\mathbb{P}}(-4) \stackrel{\omega}{\longrightarrow} \mathcal{O}_{\mathbb{P}}(-3) \stackrel{\wedge^3\omega}{\longrightarrow} \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{Pf_4} \longrightarrow 0.$$

This implies that the relative canonical bundle of $Pf_4 \subset \mathbb{P}$ is the restriction of $\mathcal{O}(-7)$, and this remains true by taking a linear section. So the canonical bundle of X_L must be trivial. Using the resolution of $\mathcal{O}_{X_L}$ obtain by restricting to L the previous complex, one computes that $h^1(\mathcal{O}_{X_L}) = 0$, and this is enough to ensure that X_L is indeed Calabi-Yau.

The same argument as in the proof of Theorem 2.3 implies that X_L and Y_L are not birationally equivalent. Proving that they are derived equivalent requires more sophisticated techniques. □

Remark. The Calabi-Yau threefolds in the G_2-Grassmannians that appear in the proof of Theorem 2.6 are degenerations of the Pfaffian-Grassmannians Calabi-Yau threefolds.

All these examples are in fact instances of Kuznetsov's Homological Projective duality, which provides a way, under favourable circumstances, to compare the derived category of a variety $X \subset \mathbb{P}(V)$, and of its linear sections, with the derived category of the dual $X^* \subset \mathbb{P}(V^\vee)$, and of its dual linear sections (see [Th18] for an introduction).

Another example with a similar flavor is that of the spinor variety $\mathbb{S}_{10} \subset \mathbb{P}(\Delta_{16})$. As we already mentionned, this variety is a very similar to $G(2,5)$. In particular it is projectively self-dual (non canonically), and its complement is an orbit of $Spin_{10}$. Moreover it is a Mukai variety, so a smooth codimension eight linear section $X_M = \mathbb{S}_{10} \cap M$ is a K3 surface of degree 12. Moreover $M^\perp$ also has codimension eight and we get another K3 surface $Y_M = \mathbb{S}_{10}^* \cap M^\perp$ in the dual projective space. These two K3 surfaces are Fourier-Mukai partners (they are derived equivalent), they are not isomorphic, and $([X_M] - [Y_M]) \times \mathbb{L} = 0$ in the Grothendieck ring of varieties [IMOU16].

4. Orbital degeneracy loci

Instead of zero loci of sections of vector bundles, it is very natural to consider morphisms between vector bundles and their degeneraly loci, where the morphism drops rank. More generally, one may consider *orbital degeneracy loci* and try to construct interesting varieties from those.

4.1. **Usual degeneracy loci.** Given a morphism $\phi : E \longrightarrow F$ between vector bundles on a variety X, the degeneracy loci are the closed subvarieties where the rank drops:

$$D_k(\phi) := \{x \in X,\ rk(\phi_x : E_x \longrightarrow F_x) \leq k\}.$$

Writing locally ϕ as a matrix of regular functions and taking $(k+1)$-minors allow to give to $D_k(\phi)$ a canonical scheme structure. If ϕ is sufficiently regular, $D_k(\phi)$ has codimension $(e-k)(f-k)$ (where e and f are the ranks of the bundles E and F), and its singular locus is exactly $D_{k+1}(\phi)$. If moreover the latter is empty, the kernel and cokernel of ϕ give well-defined vector bundles K and C on $D_k(\phi)$, with a long exact sequence

$$0 \longrightarrow K \longrightarrow E_{D_k(\phi)} \longrightarrow F_{D_k(\phi)} \longrightarrow C \longrightarrow 0.$$

A local study allows to check that the normal bundle $N_{D_k(\phi)/X} \simeq Hom(K, C)$. Using the previous exact sequence, we can deduce in the square format $e = f$ that the relative canonical bundle is

$$\omega_{D_k(\phi)/X} \simeq (\det E^\vee)^k \otimes (\det F)^k.$$

This gives a nice way to construct Calabi-Yau manifolds of small dimension: start from a Fano variety X such that $\omega_X = L^{-k}$ for some ample line bundle L; find vector bundles E, F of the same rank e such that $Hom(E, F)$ is globally generated and $(\det E^\vee) \otimes (\det F) = L$. If the dimension of X is $n < (e-k+1)^2$, a general element $\phi \in Hom(E, F)$ defines a smooth manifold $D_k(\phi)$ with trivial canonical bundle (most often a Calabi-Yau), of dimension $n - (e-k)^2$. Similar remarks apply to the case of skew-symmetric morphisms $\phi : F^\vee \longrightarrow F$ and their associated Pfaffian loci.

Example 5. Consider a three-form $\omega \in \wedge^3(\mathbb{C}^n)^\vee$, and recall that it defines a global section of the bundle $\wedge^2 Q^\vee(1)$ on $\mathbb{P}^{n-1}$. Locally this is just a family of two-forms on the quotient bundle, and we get a stratification of $\mathbb{P}^{n-1}$ by Pfaffian loci.

4.2. **General theory.** Determinantal and Pfaffian loci are special cases of a more general construction, that of orbital degeneracy loci. Suppose for example that E is a vector bundle on some variety X, with rank e. Consider a closed subvariety Y of $\wedge^m \mathbb{C}^e$, stable under the action of GL_e. Let s be a section of $\wedge^m E$. Then the locus of points $x \in X$ for which $s(x) \in \wedge^m E_x \simeq \wedge^m \mathbb{C}^e$ belongs to Y is well-defined, since the latter identification, although non canonical, only varies by an element of GL_e. This locus is precisely the orbital degeneracy locus $D_Y(s)$.

Basic properties of these loci are established in [BFMT17]. For instance, for a general section s one has

$$\mathrm{codim}(D_Y(s), X) = \mathrm{codim}(Y, \wedge^m \mathbb{C}^e),$$

$$\mathrm{Sing}(D_Y(s)) = D_{Sing(Y)}(s).$$

4.3. **Applications: Coble cubics and generalized Kummers.** For a nice example of these constructions, start with a general three-form $\omega \in \wedge^3(\mathbb{C}^9)^\vee$. The quotient bundle Q on $\mathbb{P}^8$ has rank 8, and a skew-symmetric form in eight variables has rank six in codimension 1, four in codimension 6, two in codimension 15. So the Pfaffian stratification reduces to $\mathbb{P}^8 \supset C \supset A$, where:

1. The hypersurface C is given by the Pfaffian of the induced section of $\wedge^2 Q^\vee(1)$, which is a degree four polynomial, hence a section of the line bundle $\wedge^8 Q^\vee(4) = \mathcal{O}(3) \subset Sym^4(\wedge^2 Q^\vee(1))$; so C is in fact a cubic hypersurface.
2. The singular locus of C is the smooth surface A, whose canonical bundle is trivial. This is in fact an *abelian* surface, and C is the *Coble cubic* of A, the unique cubic hypersurface which is singular exactly along A [GSW13].

On the dual projective space, denote by H the tautological hyperplane bundle, of rank 8. Then ω defines a global section of $\wedge^3 H^\vee$, and therefore, orbital degeneracy loci $D_Y(\omega)$ for each orbit closure $Y \subset \wedge^3 \mathbb{C}^8$. The codimension four orbit closure

Y_4 has particular interest [BMT19]. It can be characterized as the image of the birational Kempf collapsing

$$\begin{array}{ccc} Tot(E) & \longrightarrow & Y_4 \subset \wedge^3\mathbb{C}^8 \\ \downarrow & & \\ F(2,5,8) & & \end{array}$$

where E is the rank 31 vector bundle with fiber $\wedge^3 U_5 + U_2 \wedge U_5 \wedge \mathbb{C}^8$ over the partial flag $U_2 \subset U_5 \subset \mathbb{C}^8$.

Theorem 4.1. *The Kempf resolution of the orbital degeneracy locus $D_{Y_4}(\omega)$ is a hyperKähler fourfold.*

More precisely, this hyperKähler fourfold is isomorphic with the generalized Kummer fourfold of the abelian surface A.

Generalized Kummers of abelian surfaces were first constructed by Beauville: start with an abelian surface A and the Hilbert scheme $Hilb^n(A)$ of n-points on A; then the sum map $A^n \to A$ descends to $Hilb^n(A) \to A$, and the n-th generalized Kummer variety of A is any fiber. The resolution in Theorem 4.1 is exactly the restriction to the Kummer variety of the Hilbert-Chow map from $Hilb^3(A)$ to $Sym^3(A)$.

Generalized Kummer varieties are hyperKähler manifolds, like Hilbert schemes of points on K3 surfaces. But contrary to the latter case (for which we have e.g. Fano varieties of lines on cubic fourfolds), no locally complete projective deformation of generalized Kummers is known!

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