Three lectures on the Tits-Freudenthal square

Lecture 1

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Overview

Lie algebras are infinitesimal versions of continuous groups, and are ubiquitous in all sorts of problems with continuous groups of symmetries, in mathematics as well as in physics.

Over the complex numbers, the classification of simple Lie algebras by E. Cartan and W. Killing is one of the most spectacular achievements of 19th century mathematics. Apart from the series of classical Lie algebras, this classification exhibits five exceptional Lie algebras.

Are they the Ugly Duckling of the classification? Should we better forget them? Or, on the contrary, are these unexpected symmetry groups the source of some interesting and unexpected pieces of mathematics? This is what I believe in, and I plan to illustrate this perspective inside complex projective geometry. So the goals will be:

1. Explain the algebraic construction of exceptional Lie algebras from a pair of normed algebras, this is the Tits-Freudenthal magic square.
2. Describe a geometric version of this magic square, and a couple of related remarkable varieties.
3. Explain how they are related to some fundamental problems and conjectures.

A little bit of History

- Felix Klein (1872): Erlangen's program: classify geometries by their symmetries.
- Sophus Lie (1883): continuous transformations groups and their linearization.
- Wilhelm Killing (1888-1890): classification of simple Lie algebras.
- Elie Cartan (doctoral dissertation, 1894).

The Cartan-Killing classification. Over the complex numbers, there exist three (or four, or two, or one?) series of simple classical Lie algebras:

$$\mathfrak{sl}_n, \quad \mathfrak{so}_n, \quad \mathfrak{sp}_{2n},$$

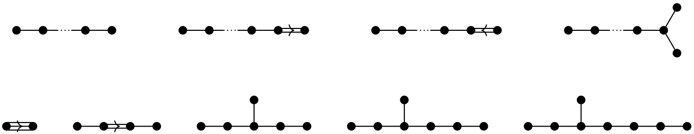
and five *exceptional Lie algebras* of dimensions $14, 52, 78, 133, 248$:

$$\mathfrak{g}_2, \quad \mathfrak{f}_4, \quad \mathfrak{e}_6, \quad \mathfrak{e}_7, \quad \mathfrak{e}_8.$$

$\longrightarrow$ real forms, classifications over all sorts of fields, symmetric spaces, graded Lie algebras, representation theory, spinors and quantum mechanics, Kac-Moody Lie algebras, finite groups of Lie type, etc.

What are the exceptional Lie algebras?

Exceptional Lie algebras are encoded in their Dynkin diagrams, that encore their root systems, from which their structure can be reconstructed:



If you do not know about root systems and so on, the simplest way to define them is to provide *models* (here $\mathbb{Z}_2$-gradings):

$$\begin{array}{cccccc} \mathfrak{g}_2 & = & \mathfrak{sl}_3 & \oplus & S^2\mathbb{C}^3, \\ \mathfrak{f}_4 & = & \mathfrak{so}_9 & \oplus & \Delta, \\ \mathfrak{e}_6 & = & \mathfrak{sl}_2 \times \mathfrak{sl}_6 & \oplus & \mathbb{C}^2 \otimes \wedge^3\mathbb{C}^6, \\ \mathfrak{e}_7 & = & \mathfrak{sl}_8 & \oplus & \wedge^4\mathbb{C}^8, \\ \mathfrak{e}_8 & = & \mathfrak{so}_{16} & \oplus & \Delta_+. \end{array}$$

Quaternions and Octonions

Obvious questions: what are really these Lie algebras? where do they come from? what are they good for? Even $\mathfrak{g}_2$ was mysterious, and Killing thought he had found two different copies!

Quaternions. Discovered by Hamilton in 1843:

$$i^2 = j^2 = k^2 = ijk = -1.$$

$\longrightarrow$ four dimensional algebra over $\mathbb{R}$, normed, associative but not commutative. A quaternion $q \in \mathbb{H}$ is a linear combination

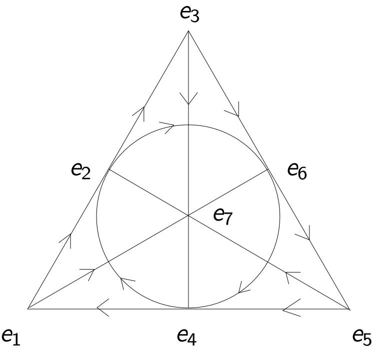
$$q = x + yi + zj + tk, \qquad x, y, z, t \in \mathbb{R}.$$

Or course $\mathbb{H} \supset \mathbb{C}$, conjugation extends: $\overline{q} = x - yi - zj - tk$, as well as the norm $|q|^2 = q\overline{q} = \overline{q}q$. The norm is multiplicative: $|pq| = |p| \times |q|$. Since $k = ij$ one can write $q = (x + yi) + (z + ti)j$ and see $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ with multiplication

$$(a, b)(c, d) = (ac - \overline{d}b, b\overline{c} + da).$$

Cayley-Dickson doubling process $\rightsquigarrow$ can we double $\mathbb{H}$?

Octonions. Discovered by Graves in 1843 and Cayley in 1845: one gets $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \longrightarrow$ eight dimensional algebra over $\mathbb{R}$, normed, non commutative, non associative but alternative.



The multiplication table of the octonions is a Fano plane, with seven points and seven lines encoding the multiplication (with $e_0 = 1$ the unit).

One can try to double again but the process stops (one needs associativity for the double algebra to be normed).

Theorem (Hurwitz 1898)

There exist only four real normed algebras (i.e. with a scalar product which is multiplicative):

$$\mathbb{A} = \mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{O}.$$

Consider the automorphism groups

$$Aut(\mathbb{A}) = \{U \in SO(\mathbb{A}),\ U(xy) = U(x)U(y)\ \ \forall x, y \in \mathbb{A}\}.$$

Theorem (Cartan 1914)

$Aut(\mathbb{O}) = G_2$ *and* $der(\mathbb{O}) = \mathfrak{g}_2$ *is its Lie algebra.*

$$Aut(\mathbb{R}) = 1, \quad Aut(\mathbb{C}) = \mathbb{Z}_2, \quad Aut(\mathbb{H}) = SO_3,$$
$$der(\mathbb{R}) = 0, \quad der(\mathbb{C}) = 0, \quad der(\mathbb{H}) = \mathfrak{so}_3.$$

Spin and triality

Triality groups generalize automorphism groups:

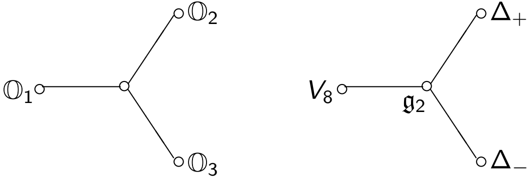
$$Tri(\mathbb{A}) \quad = \quad \{(U_1, U_2, U_3) \in SO(\mathbb{A})^3,\ U_1(xy) = U_2(x)U_3(y)\ \ \forall x, y \in \mathbb{A}\}.$$

The maps $\pi_i : Tri(\mathbb{A}) \longrightarrow SO(\mathbb{A})$ give three representations.

Theorem (Cartan 1925)

The group $Tri(\mathbb{O}) \simeq \mathrm{Spin}_8$. Each projection π_i is a twofold cover of SO_8. The 3 corresponding 8-dim'l representations of Spin_8 are inequivalent.

The Dynkin diagram D_4 is the only one with a threefold symmetry.



The triality Lie algebras $tri(\mathbb{A}) = 0,\ \mathfrak{ab}_2,\ \mathfrak{sl}_2^3,\ \mathfrak{so}_8$.

Jordan algebras

Next step: in the 1950's, when Chevalley and Schafer considered

$$H_3(\mathbb{O}) := \left\{ \begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x_3} & r_2 & x_1 \\ \overline{x_2} & \overline{x_1} & r_3 \end{pmatrix}, \quad r_i \in \mathbb{R},\ x_j \in \mathbb{O} \right\}.$$

The product $A.B = \frac{1}{2}(AB + BA)$ is commutative but non associative. The characteristic identity of *Jordan algebras* holds: $A^2(AB) = A(A^2B)$.

Jordan algebras were introduced in the 1930's by Jordan, von Neumann and Wigner as a natural mathematical framework for quantum theory.

Any associative algebra is Jordan for the product $a.b = \frac{1}{2}(ab + ba)$. A Jordan algebra is *exceptional* if it cannot be embedded as a Jordan subalgebra of an associative algebra. $H_3(\mathbb{O})$ is exceptional.

Theorem (Chevalley-Schafer 1950)

$Aut(H_3(\mathbb{O})) = F_4$, with Lie algebra $der(H_3(\mathbb{O})) = \mathfrak{f}_4$.

The Tits-Freudenhal magic square

Now consider a pair $(\mathbb{A}, \mathbb{B})$ of normed algebras and define

$$\mathfrak{g}(\mathbb{A}, \mathbb{B}) := \mathrm{d}er\mathbb{A} \times \mathrm{d}erH_3(\mathbb{B}) \oplus (\mathrm{I}m\mathbb{A} \otimes H_3(\mathbb{B})_0),$$

where $H_3(\mathbb{B})_0 \subset H_3(\mathbb{B})$ is the hyperplane of traceless matrices.

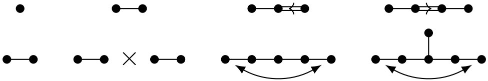
One can define a Lie algebra structure on $\mathfrak{g}(\mathbb{A}, \mathbb{B})$, with $\mathrm{der}\mathbb{A} \times \mathrm{der}H_3(\mathbb{B})$ a Lie subalgebra acting on $\mathrm{Im}\mathbb{A} \otimes H_3(\mathbb{B})_0$ in a natural way. The result of this construction is the *Freudenthal-Tits magic square of Lie algebras*:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{sl}_2$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

$\rightsquigarrow$ uniform description of the exceptional Lie algebras.

Folding

Note the first two lines are related, as visible on the Dynkin diagrams:



In terms of the cubic Jordan algebras $H_3(\mathbb{A})$, one can define a determinant Det_a, where $a = dim_{\mathbb{R}}\mathbb{A}$, and then the groups

$$\begin{array}{rcl} SL_3(\mathbb{A}) & := & \{g \in SL(H_3(\mathbb{A})),\ g^* Det_a = Det_a\}, \\ SO_3(\mathbb{A}) & := & \{g \in SL_3(\mathbb{A}),\ g(I) = I\}. \end{array}$$

This last group preserves the quadratic from $Q = \partial_I Det_a$.
The Lie algebras $\mathfrak{so}_3(\mathbb{A}) \simeq \mathfrak{g}(\mathbb{A}, \mathbb{R}) \subset \mathfrak{sl}_3(\mathbb{A}) \simeq \mathfrak{g}(\mathbb{A}, \mathbb{C})$. There is a decomposition

$$\mathfrak{sl}_3(\mathbb{A}) = \mathfrak{so}_3(\mathbb{A}) \oplus H_3(\mathbb{A})_0,$$

corresponding to a Lie algebra symmetry or $\mathbb{Z}_2$-grading.

A variant with triality

It is amazing that $\mathfrak{g}(\mathbb{A}, \mathbb{B})$ has a natural Lie algebra structure, but also the symmetry of the square appears to be miraculous.

A more symmetric variant of the Tits-Freudenthal construction was found by Vinberg in 1966, and rediscovered by several people (Allison, Dadok-Harvey, Barton-Sudbery, Landsberg-M.).

Let $(\mathbb{A}, \mathbb{B})$ be normed algebras, with the three actions of their triality Lie algebras $tri(\mathbb{A})$ and $tri(\mathbb{B})$. There is a natural Lie algebra structure on

$$\mathfrak{g}(\mathbb{A}, \mathbb{B}) = tri(\mathbb{A}) \times tri(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3).$$

Of course $tri(\mathbb{A}) \times tri(\mathbb{B})$ is a Lie subalgebra, acting on $\mathbb{A}_i \otimes \mathbb{B}_i$ in the natural way. The bracket of $a_1 \otimes b_1 \in \mathbb{A}_1 \otimes \mathbb{B}_1$ with $a_2 \otimes b_2 \in \mathbb{A}_2 \otimes \mathbb{B}_2$ is simply $a_1 a_2 \otimes b_1 b_2$, considered as an element of $\mathbb{A}_3 \otimes \mathbb{B}_3$.

$\rightsquigarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ grading on $\mathfrak{g}(\mathbb{A}, \mathbb{B})$, starting from

$$\mathfrak{g}(\mathbb{A}, \mathbb{R}) = tri(\mathbb{A}) \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3.$$

Dimension formulas

Let $a = \dim(\mathbb{A})$ and $b = \dim(\mathbb{B})$, so $a, b \in \{1, 2, 4, 8\}$.

$$\dim \mathfrak{g}(\mathbb{A}, \mathbb{B}) = \mathbf{3}\ \frac{(ab + 2a + 2b)(ab + 4a + 4b - 4)}{(a + 4)(b + 4)}.$$

Why is this formula true?? First discovered by Vogel for the exceptional series $\mathbb{B} = \mathbb{O}$, using diagrammatic/categorical methods:

$$\begin{array}{ccccccccccc} der(\mathbb{O}) & \subset & tri(\mathbb{O}) & \subset & \mathfrak{g}(\mathbb{R}, \mathbb{O}) & \subset & \mathfrak{g}(\mathbb{C}, \mathbb{O}) & \subset & \mathfrak{g}(\mathbb{H}, \mathbb{O}) & \subset & \mathfrak{g}(\mathbb{O}, \mathbb{O}) \\ || & & || & & || & & || & & || & & || \\ \mathfrak{g}_2 & & \mathfrak{so}_8 & & \mathfrak{f}_4 & & \mathfrak{e}_6 & & \mathfrak{e}_7 & & \mathfrak{e}_8 \end{array}$$

Let $a = -1$ for $\mathfrak{sl}_3$, $a = -\frac{2}{3}$ for $\mathfrak{g}_2$, $a = 0$ for $\mathfrak{so}_8$, then (Deligne-Vogel)

$$\dim \mathfrak{g}(\mathbb{A}, \mathbb{O}) = 2\ \frac{(3a + 7)(5a + 8)}{(a + 4)}.$$

$\rightsquigarrow$ Deligne computed many other more complicated dimension formulas for components in $\wedge^k \mathfrak{g}(\mathbb{A}, \mathbb{O})$, $S^k \mathfrak{g}(\mathbb{A}, \mathbb{O})$, etc.

Is there a universal Lie algebra?

Could be the top of an iceberg?
Vogel discovered that one can associate to any simple Lie algebra $\mathfrak{g}$ a triple of numbers (α, β, γ) (up to order and scale), such that

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

where $t = \alpha + \beta + \gamma$.
They are interpreted as eigenvalues of a Casimir operator.
Then all the parameters of the complex simple Lie algebras belong to only three lines:

$$\begin{array}{cc} SL & \alpha + \beta = 0, \\ OSP & \alpha + 2\beta = 0, \\ EXC & 2\alpha + 2\beta = \gamma. \end{array}$$

$\rightsquigarrow$ uniform dimension formulas, uniform decompositions into irreducibles
$\rightsquigarrow$ categorical interpretations? universal Lie algebra??
$\rightsquigarrow$ **uniform geometries**!!

Remark. The magic square can partly be extended to higher rank:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}_n$	$\mathfrak{sl}_n$	$\mathfrak{sp}_{2n}$
$\mathbb{C}$	$\mathfrak{sl}_n$	$\mathfrak{sl}_n \times \mathfrak{sl}_n$	$\mathfrak{sl}_{2n}$
$\mathbb{H}$	$\mathfrak{sp}_{2n}$	$\mathfrak{sl}_{2n}$	$\mathfrak{so}_{4n}$

$\rightsquigarrow$ Jordan algebras of higher rank,
$\rightsquigarrow$ uniform geometries of classical types,
$\rightsquigarrow$ classification of Scorza varieties (Zak).

But the exceptional line coming from the octonions is the most interesting one!

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Lecture 2

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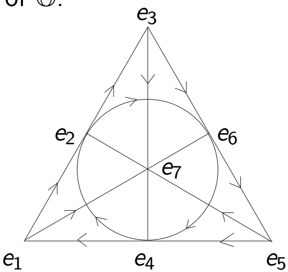
Starter: the geometry of G_2

Recall the multiplication table of $\mathbb{O}$:



You can choose $e_1 =$ any vector of unit norm, then $e_2 =$ any vector of unit norm and orthogonal to e_1. Then let $e_3 = e_1 e_2$, it has unit norm and is orthogonal to e_1, e_2. Finally choose e_4 of unit norm, orthogonal to e_1, e_2, e_3, and let $e_5 = e_4 e_1$, $e_6 = e_4 e_2$, $e_7 = e_4 e_3$: you get the correct multiplication table.

Consequence: Over $\mathbb{R}$, G_2 acts transitively on vectors of unit norm, hence on lines (and also on planes). In fact

$$S^6 \simeq G_2(\mathbb{R})/SU_3.$$

Now we switch to $\mathbb{C}$ and still denote $\mathbb{O} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, $G_2 = G_2(\mathbb{C})$.

$\rightsquigarrow$ There exist octonions of norm zero! 2-dim'l spaces of such octonions!! Even worse, planes on which all the octonionic products are zero!!!! We call them *null-planes*. They are parametrized by a closed subvariety $G_2Gr(2,7) \subset Gr(2,7)$, with transitive action of $G_2 \rightsquigarrow$ incidence diagram:

$$\begin{tikzcd}
 & G_2/B \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"] & \\
\mathbb{P}(Im\mathbb{O}) \supset \mathbb{Q}^5 = G_2/P_1 & & G_2/P_2 = G_2Gr(2,7) \subset \mathbb{P}(\mathfrak{g}_2)
\end{tikzcd}$$

$\rightsquigarrow$ subdiagram of the familiar incidence diagram for lines and planes:

$$\begin{tikzcd}
 & Fl(1,2,7) \arrow[dl, "\mathbb{P}^5"'] \arrow[dr, "\mathbb{P}^1"] & \\
\mathbb{P}(Im\mathbb{O}) = \mathbb{P}^6 & & Gr(2,7) \subset \mathbb{P}(\wedge^2 Im\mathbb{O}) = \mathbb{P}(\mathfrak{so}_7)
\end{tikzcd}$$

$\rightsquigarrow$ one-dim'l family of null-planes through a given isotropic line $\simeq v_3(\mathbb{P}^1)$.

This already contains a lot of information about $\mathfrak{g}_2$:

1. $Im\mathbb{O}$ and $\mathfrak{g}_2$ are the two fundamental representations.
2. $\mathbb{Q}^5$ and $G_2Gr(2,7)$ are the two generalized Grassmannians: the two minimal orbits in the (projectivized) fundamental representations.
3. A kind of ID for $\mathfrak{g}_2$! We can recover $G_2 = Aut(G_2Gr(2,7))$.
4. $\mathbb{Q}^5$ and $G_2Gr(2,7)$ have same dimension and Betti numbers as $\mathbb{P}^5$. In particular $Pic = \mathbb{Z}H$ but with $H^5 = 2$, resp. $H^5 = 18$, $H^5 = 1$.
5. $\mathbb{Q}^5$ and $G_2Gr(2,7)$ are both Fano, of index 5, resp. 3. By adjunction, a codimension three linear section of $G_2Gr(2,7)$ is a surface with trivial canonical bundle, in fact a K3 surface.

Theorem (Mukai)

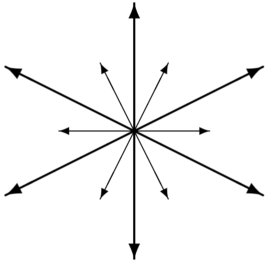
A general polarized $K3$ surface of degree 18 can be obtained like this.

6. $G_2Gr(2,7) \subset \mathbb{P}(\mathfrak{g}_2)$ is an example of *adjoint variety*, a closed $Aut(\mathfrak{g})$-orbit inside $\mathbb{P}(\mathfrak{g})$ for $\mathfrak{g}$ a simple complex Lie algebra.

Over $\mathbb{R}$, we have seen that $G_2(\mathbb{R})$ acts transitively on imaginary octonions of norm one, and $S^6 \simeq G_2(\mathbb{R})/SU_3$. Over $\mathbb{C}$, this cannot be true since there exist octonions of norm zero, but

$$\mathbb{P}(Im\mathbb{O}) - \mathbb{Q}^5 \simeq G_2/SL_3.$$

$\rightsquigarrow \mathbb{P}^6 = \mathbb{P}(Im\mathbb{O})$ compactifies the homogeneous space G_2/SL_3.
The embedding of SL_3 in G_2 is given by the long roots of $\mathfrak{g}_2$:



This gives a geometric model for the isomorphism $\mathfrak{g}_2 \simeq \mathfrak{sl}_3 \oplus S^2\mathbb{C}^3$.
$\rightsquigarrow$ In the sequel, we will give geometric incarnations of the algebraic models provided by the magic square.

The Abuaf-Ueda flop

Let us come back to the fundamental incidence diagram

$$\begin{tikzcd}
 & G_2/B \arrow[dl, "\mathbb{P}^1"', "p_1"] \arrow[dr, "\mathbb{P}^1", "p_2"'] & \\
\mathbb{P}(Im\mathbb{O}) \supset \mathbb{Q}^5 = G_2/P_1 & & G_2/P_2 = G_2Gr(2,7) \subset \mathbb{P}(\mathfrak{g}_2)
\end{tikzcd}$$

The Picard group of G_2/B is generated by pull-backs of line bundles from both sides $\rightsquigarrow$ minimal ample line bundle $\mathcal{L} = \mathcal{O}(1,1)$. We have

$$p_{1*}\mathcal{L} = C^\vee(1), \qquad p_{2*}\mathcal{L} = N^\vee(1).$$

In general, if $E \to X$ is a vector bundle, then $\mathcal{O}_E(-1) \stackrel{p}{\to} \mathbb{P}(E)$ is such that $Tot(\mathcal{O}_E(-1)) \to Tot(E)$ is the blow-up of the zero section. Here we get the *Abuaf-Ueda flop*

$$\begin{tikzcd}
 & Tot(\mathcal{L}^\vee) \arrow[dl] \arrow[dr] & \\
Tot(C(-1)) & & Tot(N(-1)) \arrow[ll, dashed, leftrightarrow, "flop"']
\end{tikzcd}$$

More concretely, consider a general section s of $\mathcal{L}$ over G_2/B. If defines a section s_1 of $C^\vee(1)$ over G_2/P_1 and a section s_2 of $N^\vee(1)$ over G_2/P_2. $\rightsquigarrow$ $Z(s_1)$ and $Z(s_2)$ are two CY-threefolds and there is a diagram

$$\begin{tikzcd} & Z(s) \arrow[dl, "Bl_{Z(s_1)}"'] \arrow[dr, "Bl_{Z(s_2)}"] & \\ G_2/P_1 & & G_2/P_2 \end{tikzcd}$$

FACTS.

- The Abuaf-Ueda flop $Tot(C(-1)) \dashrightarrow Tot(N(-1))$ induces an equivalence of derived categories (Ueda 2019, Hara 2021).
- The two Calabi-Yau threefolds $Z(s_1)$ and $Z(s_2)$ are derived equivalent (Kuznetsov 2016), but not isomorphic and even not birationally equivalent.
- $Z(s_2)$ is a flat deformation of a linear section of $G(2,7)$ (Ito-Inoue-Miura 2019), for which we already have the Pfaffian-Grassmannian derived equivalence with a linear section of the Pfaffian variety $Pf_7 \subset \mathbb{P}(\wedge^2 V_7)$ (Borisov-Caldararu 2006).

- In the Grothendieck ring of varieties, $[Z(s_1)] \neq [Z(s_2)]$ but

$$([Z(s_1)] - [Z(s_2)]).\mathbb{L} = 0,$$

where $\mathbb{L}$ denotes the class of the affine line. In particular $\mathbb{L}$ is a zero-divisor (first example by Borisov 2018).

Recall the Grothendieck ring of varieties is generated by symbols $[Z]$, for Z a quasi-projective variety, with relations $[Z \setminus W] = [Z] - [W]$ when $W \subset Z$, and $[X \times Y] = [X] \times [Y]$. From the fact that $Z(s)$ is a blow-up in two ways, we get that

$$[G_2/P_1] + [Z(s_1)].\mathbb{L} = [Z(s)] = [G_2/P_2] + [Z(s_2)].\mathbb{L}.$$

Moreover, it follows from the Bruhat decomposition that, as $\mathbb{P}^5$, the two fivefolds G_2/P_1 and G_2/P_2 admit cell-decompositions with only one cell in each dimension from 0 to 5. Hence

$$[G_2/P_1] = [G_2/P_2] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5.$$

Finally $[Z(s_1)] = [Z(s_2)]$ would imply that $Z(s_1)$ and $Z(s_2)$ are stably birational, hence birational, hence isomorphic since they are CY; a contradiction since in general their Picard groups are generated by line bundles of different volumes.

The Tits-Freudenhal magic square: Geometry

Recall how we constructed the magic square: for a pair $(\mathbb{A}, \mathbb{B})$ of normed algebras we let

$$\mathfrak{g}(\mathbb{A}, \mathbb{B}) := \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3).$$

There is a natural Lie bracket on $\mathfrak{g}(\mathbb{A}, \mathbb{B})$. One obtains the magic square of Lie algebras:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

$\rightsquigarrow$ This construction has a geometric counterpart. Main features:

- each line embodies a certain type of geometry, with points, lines, etc. parametrized by homogeneous spaces,
- lines are naturally related.

A brief review of homogeneous spaces

Let G be simple complex Lie group. A G-homogeneous space is (in these lectures) a complex projective variety X with a transitive action of G. So $X \simeq G/P$ for some subgroup P which is called a *parabolic subgroup* of G.

Example. $G = PGL_{n+1}$ acts transitively on projective space $\mathbb{P}^n$. Also on the Grassmannians $G(k, n+1)$, $1 \leq k \leq n$, and the flag manifolds $Fl(k_1, \dots, k_p, n+1)$ parametrizing flags $V_{k_1} \subset \cdots \subset V_{k_p} \subset \mathbb{C}^{n+1}$ of subspaces of dimensions $k_1 < \cdots < k_p \leq n$. There are projections

$$\begin{tikzcd}
 & & Fl(k_1, \dots, k_p, n+1) \arrow[dll] \arrow[dl, dotted] \arrow[dr, dotted] \arrow[drr] & & \\
G(k_1, n+1) & {} & & {} & G(k_p, n+1)
\end{tikzcd}$$

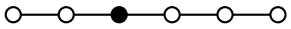
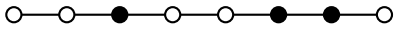
$\rightsquigarrow$ maximal G-homogeneous space $Fl(1, 2, \dots, n, n+1) = SL_{n+1}/B$, where B is the group of upper triangular matrices (a maximal solvable subgroup) + minimal G-homogeneous spaces $G(k, n+1)$.

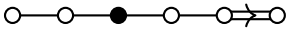
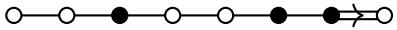
Classification of G-homogeneous spaces

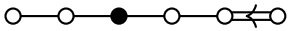
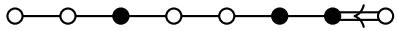
G-homogeneous spaces are classified, up to G-isomorphisms, by subsets of the Dynkin diagram Δ of G.

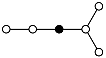
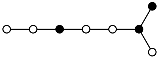
Usual notation $I \subset \Delta \rightsquigarrow G/P_I$. In particular $I = \Delta \rightsquigarrow G/B$ full flag manifold, $I = \emptyset \rightsquigarrow \{pt\}$, $I = \{i\} \rightsquigarrow G/P_i$ generalized Grassmannian.

Examples.

 $G(3,7)$,  $FI(3,6,7,9)$

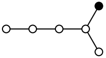
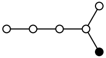
 $OG(3,13)$,  $OFI(3,6,7,17)$

 $IG(3,12)$,  $IFI(3,6,7,16)$

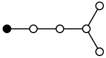
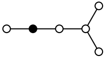
 $OG(3,12)$,  $OFI(3,6,8,16)$

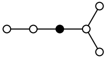
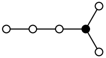
These are isotropic (or classical) Grassmannians and isotropic flag manifolds $\rightsquigarrow$ parametrize subspaces that are isotropic with respect to a non degenerate quadratic form (types B,D) or a symplectic form (type C).

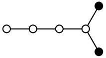
Special attention is required for *spinor varieties*, which parametrize the two families of maximal n-dimensional istropic spaces in $\mathbb{C}^{2n}$.

 $= OG(6,12)^+ \simeq OG(6,12)^- =$ 

The other G-Grassmannians for $G = SO_{12}$ are

 $= OG(1,12) = \mathbb{Q}^{10}$,  $= OG(2,12)$

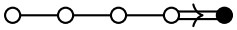
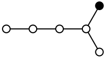
 $= OG(3,12)$,  $= OG(4,12)$.

Note that $OG(5,12) =$  is not a generalized Grassmannian!

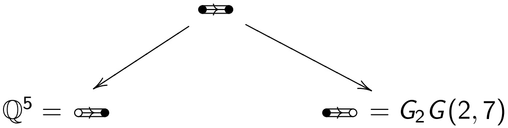
Fact: There is a G-morphism $G/P_I \rightarrow G/P_J$ if and only if $I \supset J$.

For example $OG(5,12) \rightarrow OG(6,12)^{\pm}$. In words, a five dimensional isotropic space $V_5 = V_6^+ \cap V_6^-$ for unique isotropic $V_6^{\pm}$ from $OG(6,12)^{\pm}$.

Fact: In general $Aut(G/P_I)^o \simeq G_{ad} = G/Z(G) \rightsquigarrow$ can recover $\mathfrak{g}$.

Exceptions:  $\simeq$ , $Aut = PSO_{12} \neq PSO_{11}$!

We have already seen the case of G_2: $Aut(\mathbb{Q}^5) = PSO_7 \neq G_2$!



Fact: G/P_I has a minimal projective G-embedding $G/P_I \hookrightarrow \mathbb{P}(V_{\omega_I})$, where V_{ω_I} is an irreducible G-representation (requires $\pi_1(G) = 1$). In particular the generalized Grassmannians $G/P_i \hookrightarrow \mathbb{P}(V_{\omega_i})$, where the V_{ω_i} are the *fundamental representations.* $\rightsquigarrow$ can recover the whole representation theory of G!

Example: $G = SL_{n+1}$, then $V_{\omega_i} = \wedge^i \mathbb{C}^{n+1}$ yields the Plücker embedding

$$G(i, n+1) \hookrightarrow \mathbb{P}(\wedge^i \mathbb{C}^{n+1}).$$

Example: $G = SO_m$ or Sp_m, then G/P_i is an isotropic Grassmannian and the minimal G-embedding is the restriction of the Plücker embedding:

$$\begin{tikzcd} G/P_i \arrow[r] \arrow[d] & \mathbb{P}(V_{\omega_i}) \arrow[d, "linear"] \\ G(i,m) \arrow[r] & \mathbb{P}(\wedge^i \mathbb{C}^m). \end{tikzcd}$$

Example: Caveat! This is not true for spinor varieties!

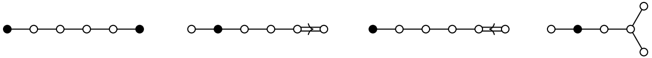
$$\begin{tikzcd} OG(n,2n)^{\pm} \arrow[r] \arrow[d] & \mathbb{P}(\Delta_{\pm}) \arrow[d, "quadratic"] \\ G(n,2n) \arrow[r] & \mathbb{P}(\wedge^n \mathbb{C}^{2n}). \end{tikzcd}$$

The representations Δ_+ and Δ_- are called *half-spin representations* of the spin group $Spin_{2n}$, the universal (degree two) cover of SO_{2n}. Their dimension is 2^{n-1}.

Conversely, given a G-representation V, the action of G on $\mathbb{P}V$ admits (at least) one orbit X of minimal dimension. Since the boundary of X must be a union of G-orbits of smaller dimension, it is empty. In other words, X is closed, hence projective and G-homogeneous. If V is irreducible, X is unique.

Example: adjoint varieties.
If $\mathfrak{g}$ is a simple Lie algebra, the adjoint action of $G = Aut(\mathfrak{g})$ on $\mathfrak{g}$ gives an irreducible representation. The unique closed orbit inside $\mathbb{P}(\mathfrak{g})$ is called the *adjoint variety* of $\mathfrak{g}$ and will be denoted G^{ad}.
For the classical types we get:

A_n $\quad B_n$ $\quad C_n$ $\quad D_n$



$Fl(1, n, n+1)$ $\quad OG(2, 2n+1)$ $\quad v_2(\mathbb{P}^{2n-1})$ $\quad OG(2, 2n)$

Two pathologies: for A_n we don't get a Grassmannian! For C_n we do, but not in the minimal embedding.

Back to the magic square.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Here is the geometric version:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$v_4(\mathbb{P}^1)$	$Fl(1,2,3)$	$IG(2,6)$	$\mathbb{OP}^2_0$
$\mathbb{C}$	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2,6)$	$\mathbb{OP}^2$
$\mathbb{H}$	$IG(3,6)$	$G(3,6)$	$OG(6,12)^+$	E_7^{hs}
$\mathbb{O}$	F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

We can even complete it to a magic triangle (Deligne-Gross):

$-\frac{2}{3}$	0	1	2	4	8
		$v_2(\mathbb{P}^1)$	$Fl(1,2,3)$	$IG(2,6)$	$\mathbb{OP}^2_0$
	3pts	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2,6)$	$\mathbb{OP}^2$
$v_3(\mathbb{P}^1)$	$(\mathbb{P}^1)^3$	$IG(3,6)$	$G(3,6)$	$OG(6,12)^+$	E_7^{hs}
G_2^{ad}	SO_8^{ad}	F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

First observations. Recall that each space $X_a^i = G/P$ from the i-th row comes with a homogeneous embedding inside $\mathbb{P}V_a^i$ for some irreducible G-representation V_a^i.

$$\dim X_a^2 = \dim X_a^1 + 1 = 2a, \quad \dim V_a^2 = \dim V_a^1 + 1 = 3a + 3,$$

$$\dim X_a^3 = 3a + 3, \quad \dim V_a^3 = 6a + 8,$$

$$\dim X_a^4 = 6a + 9, \quad \dim V_a^4 = \dim \mathfrak{g}(\mathbb{A}, \mathbb{O}) = 2\frac{(3a+7)(5a+8)}{a+4}.$$

The first two rows

Spaces from the first row are hyperplane sections of spaces from the second row. Those are projective planes over normed algebras:

$$X_a^2 \simeq \mathbb{A}\mathbb{P}^2.$$

These spaces are embedded quadratically. A dense open subset is the image of the map

$$(x, y) \in \mathbb{A}^2 \mapsto \begin{pmatrix} 1 & x & y \\ \overline{x} & x\overline{x} & y\overline{x} \\ \overline{y} & x\overline{y} & y\overline{y} \end{pmatrix} \in H_3(\mathbb{A}).$$

The space X_a^2 parametrizes rank one matrices inside $H_3(\mathbb{A})$. Summing two rank one matrices yields a rank two matrix, characterized by the vanishing of

$$Det_a(M) = \frac{1}{3}tr(M^3) - \frac{1}{2}tr(M)tr(M^2) + \frac{1}{6}tr(M)^3.$$

This formula does make sense even over $\mathbb{O}$!

Alternatively, one can think of the cubic $C_a = (Det_a = 0)$ as the projective dual hypersurface to X_a^2, parametrizing tangent hyperplanes to X_a^2 as points inside the dual projective space.
Conversely the derivatives of the determinant yield a birational map

$$\begin{tikzcd} & I_a \arrow[dl, "Bl_{\mathbb{A}\mathbb{P}^2}"'] \arrow[dr, "Bl_{\mathbb{A}\mathbb{P}^2_*}"] & \\ \mathbb{P}H_3(\mathbb{A}) \arrow[rr, dashed, "\partial Det_a"] & & \mathbb{P}H_3(\mathbb{A})^\vee \end{tikzcd}$$

The first two instances $(a = 0, 1)$ are well-known:

$$\begin{tikzcd} & I_0 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd} \qquad \begin{tikzcd} & I_1 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^5 \arrow[rr, dashed] & & \mathbb{P}^5 \end{tikzcd}$$

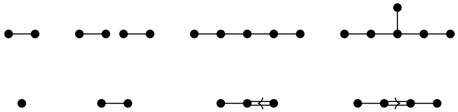
$\partial Det_0 : [r, s, t] \mapsto [st, tr, rs]$ is the classical Cremona transformation.
I_1 is the blow-up of the Veronese surface in $\mathbb{P}^5 \rightsquigarrow$ space of complete conics, instrumental in Schubert's proof that there exist exactly 3264 conics tangent to 5 given general conics.

These birational maps have very special properties:

1. Polynomials whose derivatives define birational maps are called *homaloïdal polynomials*, they are extremely special.
2. I_a contains two exceptional divisors, contracted on both sides. The complement is the open orbit of the group action.
3. A fiber over $M \in \mathbb{AP}^2_*$ is a copy of $\mathbb{P}^{a+1}$, that meets $\mathbb{AP}^2$ along a quadric $Q_M \simeq \mathbb{Q}^a$.
4. These quadrics behave like lines ($\mathbb{A}$-lines) in a plane projective geometry: in general,
   - two $\mathbb{A}$-lines meet at a single point,
   - two given points are joined by a unique $\mathbb{A}$-line.
5. The four varieties $X_a^2 = \mathbb{AP}^2$ are the four *Severi varieties*: the only smooth $X^{2m} \subset \mathbb{P}^{3m+2}$ whose secant varieties are not the full space. According to Zak, this is impossible in $\mathbb{P}^n$ for $n < 3m + 2$. $\rightsquigarrow$ Hartshorne's conjecture on complete intersections.

From $X_a^2 = \mathbb{AP}^2$ we deduce $X_a^1 = \mathbb{AP}^2_0$ by taking a hyperplane section with $\mathbb{P}H_3(\mathbb{A})_0$, the space of traceless matrices. Its automorphism group is $Aut(H_3(\mathbb{A}))$, that Identifies with $SO_3(\mathbb{A})$.

This can be interpreted as restricting a plane projective geometry to an elliptic geometry. Recall that in terms of Lie algebras, this is interpreted as *folding* the Dynkin diagram



Alternatively, this amounts to passing from $SL_3(\mathbb{A})$ to $SO_3(\mathbb{A})$, and

$$\mathfrak{sl}_3(\mathbb{A}) = \mathfrak{so}_3(\mathbb{A}) \oplus H_3(\mathbb{A})_0.$$

Remark. The quadrics that defines "lines" in $X_a^2 = \mathbb{AP}^2$ can be seen as *entry loci*: any $P \in Sec(X_a^2) - X_a^2$ belongs to infinity many secant lines, and the intersection points of these lines with X_a^2 are parametrized by a a-dimensional quadric Q_P.

Varieties of reductions

Similarly, a point $P \notin Sec(X_a^2)$ belongs to infinitely many trisecant places. Projecting from $P = I$ to traceless matrices we get infinitely many trisecant lines to $\overline{X_a^2} \subset \mathbb{P}H_3(\mathbb{A})_0$. Hence a subvariety $Z_a \subset G(2, H_3(\mathbb{A})_0)$.

Theorem (Iliev-M. 2005)

Z_a is a smooth Fano manifold of dimension $3a$, Picard number one and index $a+1$, that compactifies $\mathbb{C}^{3a}$.
The action of $SO_3(\mathbb{A})$ on Z_a has four orbits, and the open orbit

$$Z_a^0 \simeq SO_3(\mathbb{A})/T(\mathbb{A}) \rtimes \mathcal{S}_4.$$

Moreover Z_a is covered by linear space of dimension a, and there are exactly three of them passing through each point of Z_a^0. These three spaces are transverse and yield a decomposition of the tangent space as

$$\mathfrak{so}_3(\mathbb{A})/\mathfrak{t}(\mathbb{A}) \simeq \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3.$$

Hence a geometric incarnation of the triality model for $\mathfrak{g}(\mathbb{A}, \mathbb{R}) = \mathfrak{so}_3(\mathbb{R})$!

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Three lectures on the Tits-Freudenthal square

Lecture 3

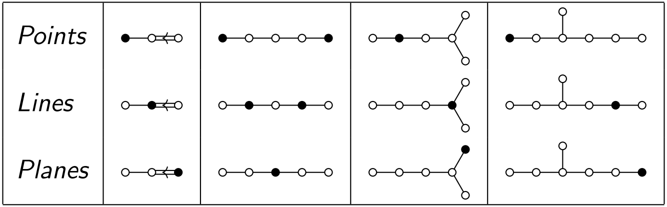
Laurent Manivel

Toulouse Mathematics Institute

December 2022

The third row of the Magic Square

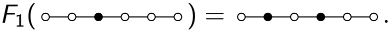
Freudenthal defined for the third line a synthetic geometry modeled on symplectic geometry in five dimensions $\rightsquigarrow$ points, isotropic lines and isotropic planes. Points are parametrized by adjoint varieties. Planes are parametrized by X_a^3 of dimension $3a+3$.



Up to isomorphisms, two different planes have only three possible relative positions. They are called *incident* when the corresponding points are joined by a line contained in X_a^3.

Intermezzo: lines in homogeneous spaces

A line in a Grassmannian $G(k,n)$ is defined by fixing spaces U_{k-1}, U_{k+1} and considering the V_k'such that $U_{k-1} \subset V_k \subset U_{k+1}$. In particular the Hilbert scheme (or Fano variety) of lines in $G(k,n)$ is homoegeneous:



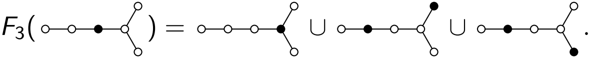
This is a general recipe:

Theorem (Landsberg -M. 2003)

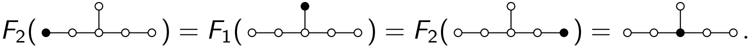
The space of lines in a generalized Grassmannian G/P_i is (in general) G-homogeneous, and its diagram is obtained by replacing the vertex i by the adjacent ones.

Beware there are some exceptions, but always true in type ADE. (In types BC one sometimes get Hilbert schemes of lines with two orbits.)

By induction, this extends to the Fano varieties F_k parametrizing copies of $\mathbb{P}^k$ inside G/P_i: one needs to isolate copies of $\bullet\!-\!\circ\!-\!\circ\!-\!\circ$.



This works specially well if we start from a homogeneous space G/P_{end} where P_{end} is defined by a vertex situated at the end of an arm of the Dynkin diagram: we can recover the G-Grassmannians encoded by vertices on the whole arm, as Hilbert schemes of linear spaces. Typically

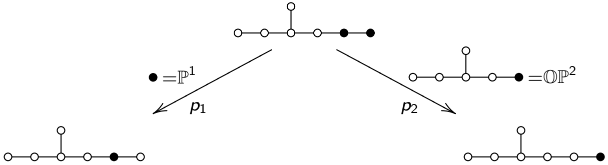


This is also true at the level of representations: starting from $G/P_{end} \subset \mathbb{P}(V_{end})$, one gets

$$G/P_k = F_{k-1}(G/P_{end}) \subset G(k, V_{end}) \subset \mathbb{P}(\wedge^k V_{end})$$

so the k-the fundamental representation is a component of $\wedge^k V_{end} \rightsquigarrow$ the most important representations are those of the form V_{end}!

More generally, there is a theory of *Tits shadows*: when $G/P_I \to G/P_J$ is a projection, ie $I \supset J$, the fiber is a homogeneous space (of a smaller semisimple group) whose diagram is obtained by suppressing the vertices that belong to J. Typically:



The fibers of p_1 are $\mathbb{P}^1$'s, mapped by p_2 to lines in $X_8^4 = E_7/P_7$.
The fibers of p_2 are copies of $\mathbb{OP}^2 = X_8^2$ parametrizing lines in E_7/P_7 passing through a given point.

What is not always (but often) true is that any copy of P_J/P_I in G/P_I is a fiber of this projection map.

Consequence

The space of lines in X_a^4 passing through a given point is a copy of X_a^2.

Conversely, we can reconstruct X_a^4 from X_a^2. Recall the latter is $\mathbb{AP}^2$, the space of rank one elements in $\mathbb{P}H_3(\mathbb{A})$. This is also the singular locus of the space of elements of rank at most two, which is the cubic $(Det_a = 0)$.

Theorem (Integrating the cubic)

$X_a^4 \subset \mathbb{P}Z(\mathbb{A})$ is the (closure of the) image of the cubic map

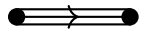
$$M \in H_3(\mathbb{A}) \mapsto [1, M, \partial Det_a(M), Det_a(M)] \in \mathbb{P}Z(\mathbb{A}),$$

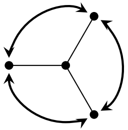
where $Z(\mathbb{A}) = \mathbb{C} \oplus H_3(\mathbb{A}) \oplus H_3(\mathbb{A})^\vee \oplus \mathbb{C}$ admits a natural symplectic structure, invariant under $Aut(X_a^4)$.

Guess the symplectic structure! And deduce that X_a^4 is *Legendrian*: each affine tangent space is a Lagrangian subspace of $Z(\mathbb{A})$. Mukai: twisted cubics over Jordan algebras.

For $a=-\frac{2}{3}$, M is just a scalar and we get a rational cubic in $\mathbb{P}^3$:

$$X^4_{-\frac{2}{3}} = v_3(\mathbb{P}^1), \qquad X^4_0 = \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1.$$





This is a geometric incarnation of folding!
Moreover the twisted cubic curve has the nice property that its tangent developpable is a quartic surface in $\mathbb{P}^3$. For $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\subset\mathbb{P}M_{2,2,2}$, where $M_{2,2,2}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ is the space of cubic matrices, we have *Cayley's hyperdeterminant*

$$\begin{aligned} HDet(A) \quad = \quad & a_{000}^2a_{111}^2+a_{001}^2a_{110}^2+a_{010}^2a_{101}^2+a_{100}^2a_{011}^2 \\ & -2a_{000}a_{001}a_{110}a_{111}-2a_{000}a_{010}a_{101}a_{111}-2a_{000}a_{011}a_{100}a_{111} \\ & -2a_{001}a_{010}a_{101}a_{110}-2a_{001}a_{011}a_{110}a_{100}-2a_{010}a_{011}a_{101}a_{100} \\ & +4a_{000}a_{011}a_{101}a_{110}+4a_{001}a_{010}a_{100}a_{111}. \end{aligned}$$

Hyperdeterminants are equations of dual varieties, but here these are the same as the tangent varieties, because of the Legendrian property.

Remarkably, these properties propagate to the whole row $X_a^4 \subset \mathbb{P}Z(\mathbb{A})$.
The tangent variety is always a quartic hypersurface, with a singular locus $W_a \supset X_a^4$ of dimension $5a + 4$.
The equation $HDet_a$ of the quartic hypersurface admits a uniform expression, independently of a.
The string

$$X_a^4 \subset W_a \subset Tan(X_a^4) \subset \mathbb{P}Z(\mathbb{A})$$

is the full stratification into orbit closures of $Aut(X_a^4)$.

Theorem

X_a^4 has the OADP property (one apparent double point property).

So a general point of $\mathbb{P}Z(\mathbb{A})$ belongs to a *unique* secant line to X_a^4
$+$ same property for tangent lines ($\simeq$ identifiability).

A flop

For $a = 2$ we can easily give representatives of the orbits:

$G(3,6)$	$e_1 \wedge e_2 \wedge e_3$
W_2	$e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5$
$TanG(3,6)$	$e_1 \wedge e_2 \wedge e_4 + e_2 \wedge e_3 \wedge e_5 + e_3 \wedge e_1 \wedge e_6$
$\mathbb{P}Z(\mathbb{C})$	$e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$

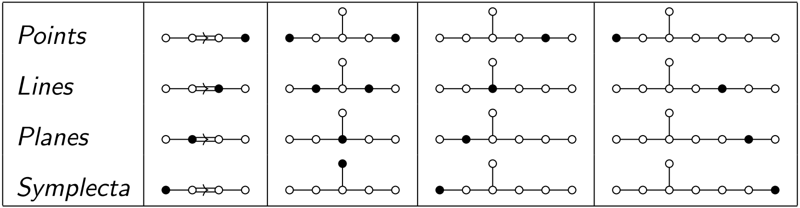
An element of W_2 determines a line and a hyperplane. Hence the following resolutions of singularities:

$$\begin{tikzcd} & \mathbb{P}(\wedge^2 \mathcal{Q}) \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbb{P}(\wedge^2 Q) \arrow[d] \arrow[dr] & Fl(1,5,6) \arrow[dl] \arrow[dr] & \mathbb{P}(\wedge^2 H) \arrow[d] \arrow[dl] \\ \mathbb{P}^5 & W_2 & \check{\mathbb{P}}^5 \end{tikzcd}$$

Question. Is the flop $\mathbb{P}(\wedge^2 Q) \dashrightarrow \mathbb{P}(\wedge^2 H)$ a derived equivalence?

The fourth row

Freudenthal's metaplectic geometries: four types of elements + incidence conditions. All modeled on the geometry of F_4 and the four F_4-Grassmannians.



By fixing a point, looking at lines-planes-symplecta through this point, and projectivizing, one obtains the symplectic geometries from the third line. In particular, the space of lines in X_a^8 passing through a given point is isomorphic to X_a^4.
Also each symplex is an adjoint variety for the Lie algebras $\mathfrak{g}(\mathbb{A}, \mathbb{H})$ of the third line.

Contact structures

Recall that an adjoint variety G^{ad} is the unique closed G-orbit inside $\mathbb{P}\mathfrak{g}$, where $\mathfrak{g}$ is a simple complex Lie algebra and $G = Aut(\mathfrak{g})$.

Example. $A_n^{ad} \simeq Fl(1, n, n+1) = (\mathbb{P}^n \times \mathbb{P}^n) \cap H$ is a variety of partial flags. At a point $f = (V_1 \subset V_n)$, the cotangent space is

$$\Omega_{Fl,f} = \{X \in \mathfrak{sl}_{n+1}, \quad X(\mathbb{C}^{n+1}) \subset V_n, \quad X(V_n) \subset V_1, \quad X(V_1) = 0\}.$$

It contains the line $L_f^\vee = \{X \in \mathfrak{sl}_{n+1},\ X(\mathbb{C}^{n+1}) \subset V_1,\ X(V_n) = 0\}$. Hence an exact sequence

$$0 \to \mathcal{H} \to T_{Fl} \to L \to 0.$$

This defines a *contact distribution* = a distribution of tangent hyperplanes in a variety X which is maximally non integrable, in the sense that the induced linear map

$$\wedge^2\mathcal{H} \hookrightarrow \wedge^2 T_X \xrightarrow{Lie} T_X \longrightarrow L$$

defines at every point a non degenerate skew-symmetric form on $\mathcal{H}$. In particular $K_X = -(n+1)L$ if X has dimension $2n+1$.

This contact structure comes from the famous Kostant-Kirillov-Souriau symplectic form on a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^\vee$. If $\lambda \in \mathcal{O}$ has stabilizer $H \subset G$, then $T_\lambda \mathcal{O} \simeq \mathfrak{g}/\mathfrak{h}$ and one can let

$$\omega_\lambda(\overline{X}, \overline{Y}) = \lambda([X, Y]).$$

This is well-defined, closed and non-degenerate: coadjoint orbits are (open) symplectic manifolds.

When $\mathfrak{g}$ is complex semisimple, then $\mathfrak{g}^\vee \simeq \mathfrak{g}$ has finitely many nilpotent orbits, which are (open) holomorphic symplectic manifolds. In particular the nilpotent cone is a symplectic singular variety $\mathcal{N}$, which can be resolved by the Springer resolution $Tot(\Omega_{G/B}) \rightarrow \mathcal{N}$.

At the other extreme there is a unique *minimal* nilpotent orbit $\mathcal{O}_{min}$, which is smooth outside the origin. This is a cone and the adjoint variety

$$G^{ad} \simeq \mathbb{P}\mathcal{O}_{min}.$$

One can show that any line is a contact line. This explains the discrepancy between

$$\dim X_a^8 = 6a + 9 \qquad \text{and} \qquad \dim Z(\mathbb{A}) = 6a + 8.$$

Summary

1. We defined the magic square of Lie algebras $\mathfrak{g}(\mathbb{A}, \mathbb{B})$. Each row can be seen as an expansion of the triality Lie algebra $\mathfrak{t}(\mathbb{A})$.
2. We defined a geometric version where each row exhibits a special kind of geometry (elliptic, plane projective, symplectic, metaplectic).
3. Elements of these geometries are parametrized by homogeneous spaces (1,1+1,3,4). In particular there is a prefered variety X_a^b, whose dimension is linear in a.
4. Remarkable properties of these spaces propagate on each row (hyperplane sections, Severi, Legendrian or contact varieties).
5. Reduction: one gets a row from the next one by looking at lines through a given point.
6. Integration: one gets a row from the previous one by a suitable rational map.

$\rightsquigarrow$ Extraspecial or typical??

VMRTs

Complex birational geometry stressed the importance of *rational curves* contained in projective manifolds: Fano manifolds always contain (many) rational curves; when a birational morphism which is not an isomorphism, it has to contract some rational curve.

$\rightsquigarrow$ It is very important to understand families $\mathcal{F}$ of rational curves, especially the covering ones, that pass through any general point.
To simplify: consider curves of minimal degree (with respect to some polarization), so that they cannot break into simpler curves.
To simplify further: try to understand the tangent directions to such curves at a general point $x \in X$.

$\rightsquigarrow$ VMRT (variety of minimal rational tangents) $\mathcal{C}_x \subset \mathbb{P}T_xX$.

In our setting, $X = G/P$ a generalized Grassmannian, $Pic(X) = \mathbb{Z}H$, the polarization H induces the minimal embedding $G/P \hookrightarrow \mathbb{P}V$, rational curves of minimal degree are *lines*, parametrized by $F_1(G/P)$ always a covering family. Moreover lines are identified with their tangents, so $X \mapsto \mathcal{C}_x$ is exactly our reduction process!!

The LeBrun-Salamon Conjecture

For any smooth variety X, the total spaces of the cotangent bundle Ω_X has a natural symplectic structure $\omega = d\theta$, where θ is the tautological Liouville-Poincaré one-form $\rightsquigarrow \mathbb{P}(\Omega_X)$ has a canonical contact structure.

Other examples: adjoint varieties $G^{ad} \subset \mathbb{P}\mathfrak{g}$.
Note that $\mathbb{P}(\Omega_{\mathbb{P}^n}) \simeq Fl(1, n, n+1)$ belongs to both types.

LeBrun-Salamon Conjecture

If a contact manifold Y is Fano, then $Y \simeq G^{ad}$ for some simple group G.

Known in small dimension, but wide open in general.

Theorem (Druel 1998)

If Y is contact Fano of dimension five and Picard number one, $Y \simeq G_2^{ad}$.

Theorem (Kebekus-Peternell-Sommese-Wisniewski 2000)

If a contact manifold Y has K_Y not nef, then either $Y \simeq \mathbb{P}(\Omega_X)$ for some variety X, or Y is Fano with Picard number one.

If Y is Fano and not a projective space, than the contact line bundle $L = TY/H$ generates the Picard group and Y is covered by *contact lines* (rational curves on which L has degree one).
Let C_y be the contact VMRT at a general point $y \in Y$.

Theorem (Kebekus 2001)

$C_y \subset \mathbb{P}(H_y)$ is smooth and Legendrian.

$\rightsquigarrow$ Can we understand Legendrian varieties? Are there many? We already know the varieties X_a^4 from the third line of the magic square (= twisted cubics over Jordan algebras).

Theorem (Legendrian reduction, Buczynski 2008)

Let V be a symplectic space, and suppose $Z \subset \mathbb{P}(V)$ is Legendrian. Let $H \subset \mathbb{P}(V)$ be a general hyperplane, with orthogonal line $h \subset H$. Then H/h is again symplectic, and the projection of $Z \cap H$ from $[h]$ to $\mathbb{P}(H/h)$ is again Legendrian.

$\rightsquigarrow$ easy construction of many smooth Legendrian varieties! Nevertheless those we are interested in are special.

Theorem (Buczynski 2006)

Let $Z \subset \mathbb{P}(V)$ be Legendrian, and cut out by quadrics. Then Z is homogeneous.

$\rightsquigarrow$ either Z is a twisted cubic over a Jordan algebra when the Picard number is one, or

$$Z = \mathbb{P}^1 \times \mathbb{Q}^{n-1} \subset \mathbb{P}^{2n+1}.$$

Idea: Z is homogeneous under G with Lie algebra

$$\mathfrak{g} \simeq I_2(Z) \subset S^2 V^\vee \simeq \mathfrak{sp}(V).$$

General principle: can recover (parts of) the Lie algebra structure from the geometry.

What about the next Legendrian varieties? Remember that X_a^4 can be constructed as the image of

$$[t, X] \mapsto [t^3, t^2 X, tcom(X), \det(X)] \in \mathbb{P}Z(\mathbb{A}).$$

The hyperplane section $X(\mathbb{A})$ obtained as the image of $H = (t^3 = det(X))$ is smooth and preserved by the action of $SL_3(\mathbb{A})$. Moreover the image of $[1, I]$ is stabilized by $SO_3(\mathbb{A})$.

Theorem (Ruzzi 2010)

The variety $X(\mathbb{A})$ is a smooth compactification of the symmetric space $SL_3(\mathbb{A})/SO_3(\mathbb{A})$, with Picard number one.

The VMRT of $X(\mathbb{A})$ is a general hyperplane section of X_a^2: a copy of X_a^1.

Example. A general hyperplane section of $X_2^4 = G(3,6)$ is a compactification $X(\mathbb{C})$ of $SL_3 \times SL_3/SL_3 \simeq SL_3$, considered as a symmetric space. A general hyperplane is $H = (f_{123} - f_{456})^\perp$ (recall the OADP property!). The complement of SL_3 is the intersection of the Grassmannian with $f_{123}^\perp \cap f_{456}^\perp \rightsquigarrow$ intersection of two Schubert divisors; singular along two copies of $\mathbb{P}^2 \times \mathbb{P}^2$!

Problem. We have seen that the reduction process (=passing to the VMRT) can be inverted in the setting of homogeneous spaces (by some birational map, explicitly given by the equations of the VMRT). Is this still true in the general setting of projective manifolds?

Rigidity

Suppose X is a Fano manifold, then $H^2(X, TX) = 0$ by Akizuki-Nakano vanishing theorem $\rightsquigarrow$ infinitesimal deformations are well-behaved. In particular, if also $H^1(X, TX) = 0$, any local deformation is constant $\rightsquigarrow$ applies to any rational homogeneous space $X = G/P$.
What about large deformations = specializations?

PROBLEM. Suppose $\mathcal{X} \to \Delta$ is a smooth projective family, with $\mathcal{X}_t \simeq X$ for all $t \neq 0$. Is $\mathcal{X}_0 \simeq X$? If yes, we say that X is *rigid*.

Theorem (Hwang-Mok)

Any rational homogeneous space $X = G/P$ is rigid, with the unique exception of $B_3/P_2 = OG(2, 7)$.

STRATEGY. Reduce to rigidity of the VMRT: X_0 must have the same VMRT as X. Then $X_0 \simeq X$. Lots of technicalities!
$\rightsquigarrow$ Series of papers from 1998 to 2005. Non ADE types are more difficult since the VMRT is not always homogeneous.

Theorem (Pasquier-Perrin 2010)

There exists a smooth family $\mathcal{X} \to \Delta$ with $\mathcal{X}_t \simeq OG(2,7)$ for all $t \neq 0$, but $\mathcal{X}_0 \neq OG(2,7)$.

In fact X_0 is obtained as the Zariski closure of projectivization of the set

$$\left\{ x + x \wedge y, \quad x, y \in Im\mathbb{O}, \quad [x,y]\,\text{null plane} \right\}$$

In particular $X_0 \supset G_2/P_1 \cup G_2/P_2$ has at least three G_2-orbits.
The complement is a $\mathbb{C}^*$-bundle over G_2/B: a *horospherical variety*.

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \arrow[rr, "\pi"] \arrow[d] & & X_0 \subset \mathbb{P}(Im\mathbb{O} \oplus \mathfrak{g}_2) \\
 & G_2/B \arrow[dl] \arrow[dr] & & \\
G_2/P_1 & & G_2/P_2 &
\end{tikzcd}$$

The morphism π blows-up the two closed orbits G_2/P_1 and G_2/P_2.
So X_0 has Picard number one. On the other hand, the action of G_2 on $OG(2,7)$ has only two orbits, G_2/P_2 and its complement!

Outside generalized Grassmannians, some results are known.

Theorem (Weber-Wisniewski 2018)

A generalized complete flag manifold G/B is always rigid.

If the Picard number is not maximal this is not true anymore.
For example $F = Fl(1,2,4)$ has a non trivial degeneration. Indeed, observe that $F = \mathbb{P}_{\mathbb{P}^3}(Q)$ and that fixing a symplectic form one gets $Q^\vee$ as an extension by $\mathcal{O}(-1)$ by the null-correlation bundle N.
$\rightsquigarrow$ F degenerates to $\mathbb{P}_{\mathbb{P}^3}(N \oplus \mathcal{O}(1))$.

Theorem (Qifeng Li 2022)

Suppose $G/B \to G/P$ is a $\mathbb{P}^1$-fibration. Then G/P is rigid, except if $G/P = Fl(1,2,4)$ or $OFl(2,3,8)$.

The answer we should expect to the following question is unclear:

Question. Which G/P's are not rigid?

Similar strategies work for a few other *quasi-homogeneous* varieties.

Odd-symplectic Grassmannians. Recall $IG(k, 2n) \subset G(k, 2n)$ is the isotropic Grassmannian parametrizing k-dim'l subspaces of $\mathbb{C}^{2n}$, isotropic w.r.t a non-degenerate symplectic form ω.

Easy fact

The VMRT is non-homogeneous for $k < n$.

Indeed a line in $G(k, 2n)$ passing through a fixed V_k is defined by a flag $U_{k-1} \subset U_{k+1}$. We need $U_{k-1} \subset V_k \subset U_{k+1}$ so if V_k is isotropic, U_{k-1} must be isotropic and $U_{k+1} \subset U_{k-1}^{\perp}$. But this does not force U_{k+1} to be isotropic!

$\rightsquigarrow$ a two orbit variety, rather than homogeneous (in fact a scroll).

$\rightsquigarrow$ Needs to adapt the argument, but works essentially the same for $IG(k, 2n+1)$.

Theorem (Park 2016)

Odd Lagrangian Grassmannnians $IG(n, 2n+1)$ are rigid.

Odd symplectic Grassmannians are specially interesting because:

- Their automorphism groups are the odd symplectic groups $Sp_{2n+1} = Sp_{2n} \rtimes \mathbb{C}^{2n}$, which are not (semi)simple but behave very much as if they were with respect to dimension or decomposition formulas (Gelfand-Zelevinsky 1984, Proctor 1988), classical and quantum cohomology (Mihai 2007, Perrin & al 2018)
$\rightsquigarrow$ nice specialization of the universal Lie algebra?
- They are two-orbits varieties, the two orbits being distinguished by the fact that an isotropic k-plane may or may not contain the kernel of the skew-form
$\rightsquigarrow$ the closed orbit in $IG(k, 2n+1)$ is simply $IG(k-1, 2n)$.

$\rightsquigarrow$ Exceptional analogues: sextonions $\mathbb{H} \subset \mathbb{S} \subset \mathbb{O}$ lead to $E_{7\frac{1}{2}}$ and singular versions of the Legendrian and Severi varieties.

Symmetric varieties. Recall the symmetric variety $X(\mathbb{A})$ is not homogeneous, but a general hyperplane section of the homogeneous space X_a^4, with VMRT X_a^1 also homogeneous.

Theorem (Kim-Park 2019, Chen-Fu-Li 2022)

The symmetric varieties $X(\mathbb{A})$ are rigid.

Surprisingly there are very few symmetric spaces of Picard number one. A trivial example is $\mathbb{P}^n$, considered as a compactification of the complement of a smooth quadric, which is $O_{n+1}/O_1 \times O_n$. Less trivial is $\mathbb{OP}^2$, seen as a compactification of $F_4/Spin_9 \rightsquigarrow$ homogeneous for a bigger group.

Theorem (Ruzzi 2010)

Suppose that X is a smooth symmetric space of Picard number one, not homogeneous. Then X is either:

1. *One of the four symmetric varieties $X(\mathbb{A})$.*
2. *A Cayley Grassmannian.*
3. *A double Cayley Grassmannian.*

Definition. Fix a general three-form $\omega \in \wedge^3 V_7^\vee$, its stabilizer is $G_2 \subset SL(V_7)$. The Cayley Grassmannian is the variety $CG \subset G(4, V_7)$ parametrizing four-planes on which ω restricts to zero.
$\rightsquigarrow$ Fano manifold of dimension 8, Picard number 1, index 4.

PROPERTIES. (M. 2016)

- CG is a compactification of $G_2/SL_2^s SL_2^\ell$, where SL_2^s and SL_2^ℓ are the two embeddings of SL_2 in G_2 defined by short and long roots.
- CG parametrizes the subalgebras of $\mathbb{O}$ isomorphic to $\mathbb{H}$, and their degenerations.
- CG is made of three G_2-orbits, the closed one being $\mathbb{Q}^5$.
- In we blow up the closed orbit, we get two invariant divisors E and F with $\mathbb{P}^2$-bundle structures:

$$\begin{array}{ccccc}
\mathbb{P}(Sym^2 C) & \simeq & E \subset \tilde{CG} \supset F & \simeq & \mathbb{P}(Sym^2 N) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}^5 & & CG & & G_2^{ad}
\end{array}$$

Here C is the so-called *Cayley bundle* on $\mathbb{Q}^5$, which is G_2-equivariant, and N is the rank two tautological bundle on $G_2^{ad} \subset G(2,7)$.

- The quantum cohomology is known (Benedetti-M. 2017).
- The derived category has a Lefschetz collection (Guseva 2022).

DEFINITION. Consider the spinor variety $OG(7,14)^+ \subset G(7,14)$. The Plucker line bundle restricts to L^2 where L generates the Picard group. This line bundle defines the spinor embedding

$$OG(7,14)^+ \hookrightarrow \mathbb{P}(\Delta_+),$$

where Δ^+ is one of the half-spin representations, of dimension 64. If U is the restriction of the tautological rank seven bundle on $G(7,14)$, it has no section but $U \otimes L$ is generated by global sections, and

$$H^0(OG(7,14)^+, U \otimes L) \simeq \Delta_-,$$

the other half-spin representation. Both Δ_+ and Δ_- have open orbits of $\mathbb{C}^* \times Spin_7$. The double Cayley Grassmannian $DG \subset OG(7,14)^+$ is defined as the zero-locus of a general section of $U \otimes L$.

$\rightsquigarrow$ Fano manifold of dimension 14, Picard number 1, index 7.

Properties. (M. 2020)

- DG is a compactification of $G_2 = G_2 \times G_2 / G_2$.
- DG parametrizes subalgebras of $\mathbb{O} \otimes \mathbb{C}$, the algebra of bioctonions.
- DG is made of three $G_2 \times G_2$-orbits, the closed one being $\mathbb{Q}^5 \times \mathbb{Q}^5$.
- In we blow up the closed orbit, we get two invariant divisors E and F with $\mathbb{P}^3$-bundle structures:

$$\begin{array}{ccccc} \mathbb{P}(C \boxtimes C) & \simeq & E \subset \tilde{DG} \supset F & \simeq & \mathbb{P}(N \boxtimes N) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}^5 \times \mathbb{Q}^5 & & DG & & G_2^{ad} \times G_2^{ad} \end{array}$$

Questions.

- What is the VMRT of DG? Is DG rigid?
- What is its (quantum) cohomology ring? Its derived category?
- What about $\mathbb{O} \otimes \mathbb{H}$ and $\mathbb{O} \otimes \mathbb{O}$?

There should exist a nice compactification of the 128-dimensional symmetric space that corresponds to the $\mathbb{Z}^2$-grading $\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \Delta_+$, with automorphism group $E_8 \rightsquigarrow$ bioctonionic plane $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$??

Tits shadows and hyperplane sections

A strong interest for homogeneous spaces comes from the fact that they are one of the most basic sources of Fano manifolds: indeed they usually have high index $\rightsquigarrow$ can cut them by linear spaces or low degree hypersurfaces, sometimes even by vector bundles, and keep the Fano property. But we loose control, for example on automorphisms.

BASIC QUESTION. Suppose $X = G/P \cap H$ is a hyperplane section of $G/P \subset \mathbb{P}(V)$. Is $Stab_G(H) \rightarrow Aut(X)$ an isomorphism?

One can try the following strategy. Suppose G/P is covered by $\mathbb{P}^m$'s, m maximal $\rightsquigarrow$ give $\mathbb{P}^{m-1}$'s in X, except when they are themselves already contained in H. Suppose G/P has the *Unique Extension Property*, meaning that any $\mathbb{P}^{m-1}$ is contained in a unique $\mathbb{P}^m$. Then:

Proposition

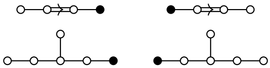
The extension morphism $F_{m-1}(X) \longrightarrow F_m(G/P)$ is the blowup of $F_m(X)$.

$\rightsquigarrow$ Since any automorphism of X extends to $F_{m-1}(X)$ and $F_m(X)$, it also extends to $F_m(G/P)$, hence comes from G (in most cases)!

Proposition

In the long case, the Unique Extension Property holds if G/P is neither an orthogonal Grassmannian, nor the adjoint variety F_4^{ad}.

But then we can use Tits shadows to describe maximal quadrics:



$$Hilb_{\mathbb{Q}^5}(F_4^{ad}) = F_4/P_4 = \mathbb{OP}^2 \cap H_0,$$
$$Hilb_{\mathbb{Q}^8}(\mathbb{OP}^2) = \mathbb{OP}^2.$$

We deduce that $F_4^{ad} = F_4/P_1$ and $F_4/P_4 = \mathbb{OP}^2 \cap H_0$ both have the Unique Extension Property for quadrics. In the end we get:

Theorem (Benedetti-M. 2022)

Any element of $Aut(X)^0$ can be lifted to $Aut(G/P)$. But if G/P is itself a hyperplane section of another G'/P', not necessarily true for $Aut(X)$!

$\rightsquigarrow$ exceptions are related to Jordan algebras and the first two lines of the magic square!

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