The Geometry of Tame Tensors

JM Invariant at 60

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A toy case: Rubik's cubes

We consider tensors of format $3 \times 3 \times 3$: tensors in $R = V_1 \otimes V_2 \otimes V_3$ where V_1, V_2, V_3 are three-dimensional vector spaces, up to the action of $G = GL(V_1) \times GL(V_2) \times GL(V_3)$.

Theorem (Ng 1995, Bhargava-Ho 2013)

There is a bijection between nondegenerate G-orbits in R and collections (C, L_1, L_2, L_3) with C a genus one curve and L_1, L_2, L_3 line bundles of degree three on C such that $L_1^{\otimes 2} = L_2 \otimes L_3$.

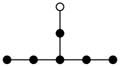
GIT quotient: consider $G_0 = SL(V_1) \times SL(V_2) \times SL(V_3)$. Then

$$R /\!/ G_0 \simeq \mathbb{A}^3$$

is an affine space. More precisely the invariant ring $R^{G_0} \simeq \mathbb{C}[I_6, I_9, I_{12}]$ is a polynomial ring over invariants of degrees $6, 9, 12$.

Vinberg and Kac

Nice behavior explained by the fact that $R = V_1 \otimes V_2 \otimes V_3$ is a *theta-representation*. Starting from affine E_6



and choosing the central vertex, Kac and Vinberg tell us we must get a $\mathbb{Z}_3$-grading of $\mathfrak{e}_6$, namely

$$\mathfrak{e}_6 = \mathfrak{sl}(V_1) \times \mathfrak{sl}(V_2) \times \mathfrak{sl}(V_3) \oplus (V_1 \otimes V_2 \otimes V_3) \oplus (V_1 \otimes V_2 \otimes V_3)^\vee$$

In this situation, the G_0-orbits in $R = V_1 \otimes V_2 \otimes V_3$ can be classified exactly as in Jordan theory. In a nutshell:

- **semisimple** and **nilpotent** elements + Jordan decompositions,
- one can define **Cartan subspaces** $\mathfrak{c} \subset R$, and every semisimple element in R is conjugate to some element of a given $\mathfrak{c}$,
- the Weyl group $W_\mathfrak{c} := N(\mathfrak{c})/Z(\mathfrak{c})$ is a **complex reflection group**, and this is why $R /\!\!/ G_0 \simeq \mathfrak{c}/W_\mathfrak{c}$ is an affine space.

Cartan

So we fix a Cartan subspace $\mathfrak{c} = \langle h_1, h_2, h_3 \rangle$ where

$$\begin{aligned} h_1 &= a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3, \\ h_2 &= a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_2, \\ h_3 &= a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_1 \otimes c_3 + a_3 \otimes b_2 \otimes c_1. \end{aligned}$$

Fact. $W_{\mathfrak{c}}$ is the complex reflection group G_{25} in the Shephard-Todd classification, of order 648. This is a degree three extension of the *Hessian group* = automorphism group of the Hesse pencil of plane cubics

$$\lambda(u^3 + v^3 + w^3) - \mu uvw = 0.$$

A generic $r \in R$ is equivalent to some $r = uh_1 + vh_2 + wh_3 \in \mathfrak{c}$. Consider

$$C_1 = \{[x] \in \mathbb{P}(V_1^\vee),\ r(x, \bullet, \bullet) \text{ degenerate}\},$$

$$C_{12} = \{([x],[y]) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee),\ r(x, y, \bullet) = 0\}$$

$\rightsquigarrow$ 6 models of the same genus one curve C.

Hesse

These 6 models are related by correspondences

$$
\begin{array}{ccccccc}
 & & & C_{12} & & & \\
 & & \swarrow & & \searrow & & \\
\mathbb{P}(V_1^\vee) \supset & C_1 & & & & C_2 & \subset \mathbb{P}(V_2^\vee) \\
 & \uparrow & & & & \uparrow & \\
\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_3^\vee) \supset & C_{13} & & & & C_{23} & \subset \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee) \\
 & & \searrow & & \swarrow & & \\
 & & & C_3 & & &
\end{array}
$$

where all the arrows are isomorphisms. (But not commutative!)
Equation of C_1: we recover the Hesse pencil

$$0 = \det \begin{pmatrix} ux_1 & vx_2 & wx_3 \\ wx_2 & ux_3 & vx_1 \\ vx_3 & wx_1 & ux_2 \end{pmatrix} = x_1x_2x_3(u^3+v^3+w^3) - (x_1^3+x_2^3+x_3^3)uvw.$$

Cayley

Among the invariants of R there is the $3 \times 3 \times 3$ hyperdeterminant, of degree 36. More complicated than Cayley's $2 \times 2 \times 2$ hyperdeterminant, of degree four; the orbit structure of $\mathbb{P}^7 \supset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is very simple:

$$\mathbb{P}^7 \supset \text{Quartic} \supset \text{Part.Decomposable} \supset \text{Decomposable}$$

So we reduce to format $2 \times 2 \times 2$ in the relative setting: r defines a global section of the rank 8 bundle $Q_1 \boxtimes Q_2 \boxtimes Q_3$ on $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$. We deduce:

- A **quadric section** $\mathcal{Q}$ of $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$, whose equation is given by the relative hyperdeterminant.
- A **surface** $\mathcal{S} \subset Sing(\mathcal{Q})$ where r becomes decomposable.

Proposition

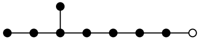
$\mathcal{S}$ is an abelian surface. In fact $\mathcal{S} \simeq C \times C$.

Coble

This is reminiscent of the classical Coble cubics (1917), whose singular loci are abelian surfaces:

$$\mathbb{P}^8 \supset \mathcal{C} \supset \mathcal{A}.$$

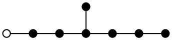
Gruson-Sam-Weyman (2013) observed that these cubics can easily be constructed from a generic tensor $r \in \wedge^3\mathbb{C}^9$, by passing to the relative setting and reducing to $\wedge^2\mathbb{C}^8$.



Similarly, the Coble quartics (1928) are singular along Kummer threefolds, which are quotients of abelian threefolds:

$$\mathbb{P}^7 \supset \mathcal{Q} \supset \mathcal{K}.$$

Again this can be recovered from a generic tensor $r \in \wedge^4\mathbb{C}^8$, by passing to the relative setting and reducing to $\wedge^3\mathbb{C}^7$.



Kummer

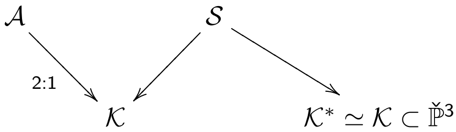
Another classical family of surfaces related to abelian varieties is the family of *Kummer surfaces*: quartic surfaces in $\mathbb{P}^3$ with 16 nodes, defined by one polynomial of type

$$A(x^2y^2+z^2t^2)+B(x^2z^2+y^2t^2)+C(x^2t^2+y^2z^2)+2Dxyzt+E(x^4+y^4+z^4+t^4)$$

for some coefficients (A, B, C, D, E) satisfying the cubic condition

$$4E^3-(A^2+B^2+C^2-D^2)E+ABC=0.$$

Such a surface $\mathcal{K}$ is a quotient of an abelian surface $\mathcal{A}$ by (-1). The minimal resolution is a K3 surface $\mathcal{S}$:



The 16 nodes in $\mathcal{K}^* \simeq \mathcal{K}$ give 16 conics in $\mathcal{K}$, called *tropes* and defining a 16_6 configuration.

Spinors

Claim. Kummer surfaces are closely related to the half-spin representations of $Spin_{10}$!

Two half-spin representations Δ_+ and Δ_-, in duality. These are *minuscule representations*.
If $\omega_\pm$ are weights of $\Delta_\pm$, $\langle\omega_+,\omega_-\rangle$ can take only three values. Say that ω_+ and ω_- are *incident* if $\langle\omega_+,\omega_-\rangle$ is not the intermediate one.

Proposition

This defines a 16_6 configuration of Kummer type.

To get something more geometric, consider the *spinor tenfolds*

$$\mathbb{S}_+ \simeq OG(5,10)^+ \subset \mathbb{P}(\Delta_+), \qquad \mathbb{S}_- \simeq OG(5,10)^- \subset \mathbb{P}(\Delta_-).$$

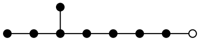
Facts.

- $\mathbb{S}_+$ and $\mathbb{S}_-$ are projectively dual, isomorphic as projective varieties
- Up to isomorphism, they admit only finitely many linear sections of codimension ≤ 3 (classified ; geometric study by Kuznetsov)

Proposition (Kuznetsov, -)

1. *There are two different types of smooth codim 2 sections of* $\mathbb{S}_+$.
2. *The special ones define a divisor* $\mathcal{R}_2 \subset G(2, \Delta_-)$, *the* **spinor quadratic line complex**.
3. *There are four different types of smooth codim 3 sections of* $\mathbb{S}_+$.
4. *The special ones define a divisor* $\mathcal{R}_3 \subset G(3, \Delta_-)$, *the* **spinor quartic plane complex**.
5. *For* $P_4 \subset \Delta_-$, *the intersection* $\mathcal{R}_3 \cap G(3, P_4)$ *is a Kummer surface.*

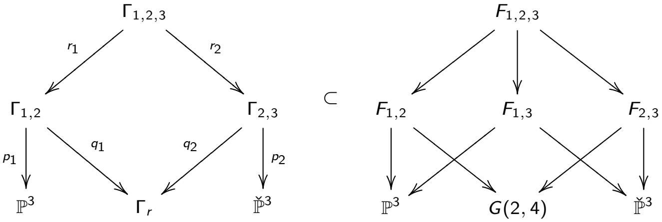
A crucial point is that $\mathbb{C}^4 \otimes \Delta_-$ is a theta-representation:



$$\mathfrak{e}_8 = \mathfrak{sl}_4 \times \mathfrak{so}_{10} \oplus (\mathbb{C}^4 \otimes \Delta_+) \oplus (\wedge^2 \mathbb{C}^4 \otimes V_{10}) \oplus (\wedge^3 \mathbb{C}^4 \otimes \Delta_-)$$

Here a Cartan subspace $\mathfrak{c}$ has rank four and $W_{\mathfrak{c}}$ is the reflexion group G_{31} in the Shephard-Todd classification: order 46080, invariants of degrees $8, 12, 20, 24$. Hence a nice GIT moduli space of codimension 4 sections.

Consider a tensor $r \in \mathbb{C}^4 \otimes \Delta_- = Hom(\mathbb{C}^4, \Delta_-)$. We can pullback $\mathcal{R}_2$ by r and get a quadratic line complex: a quadratic section Γ_r of $G(2,4)$ (GSW). There are two families of linear spaces in $G(2,4)$, hence



p_1 and p_2 are conic bundles that degenerate along K and K^* (Klein). Can also pull back to $F_{1,3}$: r defines a global section of $S^2(U_1 \wedge U_3)^\vee$ $\rightsquigarrow$ degenerates/vanishes along a hypersurface B_r/surface A_r.

Proposition

1. *B_r is a* **quadric section** *of $F_{1,3}$,*
2. *$A_r = Sing(B_r)$ is the* **abelian surface** *whose Kummer is K_r,*
3. *B_r is the* **unique** *quadric hypersurface that is singular along A_r.*

$\rightsquigarrow$ Coble type situation: a hypersurface with a special singular locus, and uniquely determined by this locus!

Alternative. Consider $OG(2,10)$ with its rank four spinor bundles $\mathcal{S}_\pm$:

$$\Gamma(OG(2,10), \mathcal{S}_\pm) = \Delta_\pm.$$

Our tensor r defines a morphism between vector bundles on $OG(2,10)$:

$$\mathbb{C}^4 \otimes \mathcal{O}_{OG(2,10)} \xrightarrow{\bar{r}} \mathcal{S}_+$$

The rank is at most i along some locus $D_i(r) \subset OG(2,10)$.

Proposition

1. *$D_3(r)$ is a quadric section of $OG(2,10)$*
2. *$D_2(r) \subset Sing(D_3(r))$ and $D_1(r) \subset Sing(D_2(r))$*
3. *$D_1(r)$ is a smooth Fano fourfold of index one*
4. *the kernel map $D_1(r) \longrightarrow \check{\mathbb{P}}^3$ is a conic bundle that degenerates along the Kummer surface K_r*

Pencils of quadrics: $\mathbb{C}^2 \otimes S^2\mathbb{C}^{2n}$

Even classical models can be interesting: consider

$$\mathfrak{sl}_{2n} = \mathfrak{so}_{2n} \oplus S^2_0\mathbb{C}^{2n}$$

We have a reference quadric Q_1, non degenerate, and we choose a second one $Q_2 = S^2_0\mathbb{C}^{2n}$ traceless. Generically

$$Q_1 = \sum_i x_i^2, \qquad Q_2 = \sum_i a_i x_i^2$$

with $a_1 + \cdots + a_{2n} = 0$: this is the Cartan subspace.

Each $OG(k, 2n)$ is stratified by the rank of Q_2:

$$D_{k-1}(Q_2) \supset \cdots \supset D_0(Q_2).$$

The hypersurface $D_{k-1}(Q_2)$ is a *quadric section* of $OG(k, 2n)$ with respect to Plücker. For $k = n$ yields **quartic sections** of the spinor varieties $\mathbb{S}_\pm = OG(n, 2n)_\pm$.

$\mathbf{n=3}$, the spinor variety is $\mathbb{P}^3$ and we recover Klein's point of view on Kummer surfaces, along with their normalized equations

$$rst(v_0^4+v_1^4+v_2^4+v_3^4)+r(s^2+t^2-w^2)(v_0^2v_1^2+v_2^2v_3^2)+$$
$$+s(t^2+r^2-v^2)(v_0^2v_2^2+v_1^2v_3^2)+t(r^2+s^2-u^2)(v_0^2v_3^2+v_1^2v_2^2)$$
$$-2(r^2s+s^2t+t^2r+uvw)v_0v_1v_2v_3=0.$$

$\mathbf{n=4}$, the spinor variety is $\mathbb{Q}^6$ (triality) and the quartic belongs to a 13-dimensional linear system generated by

$$\begin{aligned}
A_1 &= v_0v_1v_2v_3-v_4v_5v_6v_7 & A_8 &= (v_4^2+v_5^2+v_6^2+v_7^2)^2\\
A_2 &= v_0v_1v_4v_5-v_2v_3v_6v_7 & A_9 &= (v_2^2+v_3^2+v_6^2+v_7^2)^2\\
A_3 &= v_0v_1v_6v_7-v_2v_3v_4v_5 & A_{10} &= (v_2^2+v_3^2+v_4^2+v_5^2)^2\\
A_4 &= v_0v_2v_4v_6-v_1v_3v_5v_7 & A_{11} &= (v_1^2+v_3^2+v_5^2+v_7^2)^2\\
A_5 &= v_0v_2v_5v_7-v_1v_3v_4v_6 & A_{12} &= (v_1^2+v_3^2+v_4^2+v_6^2)^2\\
A_6 &= v_0v_3v_4v_7-v_1v_2v_5v_6 & A_{13} &= (v_1^2+v_2^2+v_5^2+v_6^2)^2\\
A_7 &= v_0v_3v_5v_6-v_1v_2v_4v_7 & A_{14} &= (v_1^2+v_2^2+v_4^2+v_7^2)^2
\end{aligned}$$

One recognizes the seven lines in a Fano plane!

Modular interpretations

The hypersurface D_{k-1} has multiplicity k along $D_0 = F_k(Q_1 \cap Q_2)$, which has been (and is still) studied a lot. We expect D_{k-1} is uniquely defined by D_0, we can prove it for $k \ll n$.

The pencil contains $2n$ singular quadrics and defines a *hyperelliptic curve* C of genus $g = n - 1$.
Known since Reid (1972) and Desale-Ramanan (1976) that

$$F_{n-1}(Q_1 \cap Q_2) \simeq Jac(C), \qquad F_{n-2}(Q_1 \cap Q_2) \simeq SU_C(2,1).$$

Moduli spaces in genus 2 ad 3 are also related to Coble hypersurfaces:

1. for $g = 2$, $SU_C(3) \xrightarrow{2:1} \mathbb{P}^8$ branched along the Coble sextic,
2. for $g = 3$, $SU_C(2)$ is the Coble quartic.

(GSW) observed that these moduli spaces can be recovered from the theta-representations $\wedge^3\mathbb{C}^9$ and $\wedge^4\mathbb{C}^8$.

These even moduli space are singular; what about their smooth partners, the odd moduli spaces $SU_C(r,1)$?

Four-forms

$$\mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \wedge^4 \mathbb{C}^8$$

A four-form $\omega \in \wedge^4 V_8^\vee \simeq \wedge^4 V_8$ reduces to a three-form in seven variables on the twisted quotient bundle over $\mathbb{P}(V_8)$:

$$\wedge^4 V_8^\vee \simeq H^0(\mathbb{P}(V_8), \wedge^3 Q^\vee(1)),$$

$$\omega \mapsto \tilde{\omega}_{[v]} = \omega(v, \bullet, \bullet, \bullet).$$

The degree seven invariant on $\wedge^3 \mathbb{C}^7$ defines the Coble quartic $Q(\omega)$.

Alternatively, we can reduce to a three-form in six variables over $Fl(1,7,V_8) \rightsquigarrow$ threefold $A(\omega)$ with

$$A(\omega)$$

$$\swarrow \qquad \searrow$$

$$\mathbb{P}(V_8) \supset Q(\omega) \supset K(\omega) \qquad\qquad K(\omega) \subset Q(\omega)^* \subset \mathbb{P}(V_8^\vee)$$

Same situation as for Kummer surfaces: $A(\omega) \simeq Jac(C)$ is an abelian threefold and $Q(\omega)$ selfdual (GSW-Thorne).

Even simpler over $G(2, V_8)$: two-forms in six variables!

Theorem ($g = 3$)

$SU_C(2,1)$ is the singular locus of a Coble quadric in $G(2,8)$. Moreover this Coble quadric is projectively self-dual.

Grassmann duality (Chaput, unpublished):
Let $H \subset G = G(k, k+\ell)$ be a hypersurface. Let $h = [U_k]$ be a generic point, then

$$T_hH \subset T_hG = Hom(U_k, V_{k+\ell}/U_k) \rightsquigarrow (T_hH)^\perp \subset Hom(V_{k+\ell}/U_k, U_k)$$

spanned by ϕ surjective in general (if $\ell > k$)
$\rightsquigarrow$ the kernel gives $W_\ell \supset U_k$ defining $h^* \in H^* \subset G(\ell, k+\ell)$.

Three-forms

$$\mathfrak{e}_8 = \mathfrak{sl}_9 \oplus \wedge^3\mathbb{C}^9 \oplus \wedge^6\mathbb{C}^9$$

A three-form $\sigma \in \wedge^3 V_9^\vee$ reduces to a two-form in eight variables on the twisted quotient bundle over $\mathbb{P}(V_9)$, the degree four Pfaffian defines the Coble cubic $C(\sigma)$, singular along an abelian surface $A(\sigma) \simeq Jac(C)$. The moduli space in on the dual side, where $C(\sigma)^* = S(\sigma)$ sextic in $\mathbb{P}(V_9)$ (Dolgachev, Ortega-Nguyen).

Again we can reduce to a three-form in six variables over $G(6, V_9)$. The quartic invariant gives a quadric section $Q(\sigma)$ + a locus $D(\sigma)$ where the three-form becomes completely decomposable.

Theorem ($g = 2$)

$SU_C(3,1) = D(\sigma)$ is the singular locus of the singular locus of the Coble quadric in $G(3,9)$.

The proofs of both theorems relies on the geometry of lines in the moduli spaces of vector bundles on curves.

Recap

Tame tensors = those appearing in gradings of simple Lie algebras. Classified by Vinberg theory; their GIT moduli space is just an affine space, quotient of a Cartan space $\mathfrak{c}$ by a complex reflection group $W_{\mathfrak{c}}$.

From a generic such tensor one can in many cases construct:

1. an abelian variety (curve, surface, threefold)
2. other interesting geometric loci (Kummer, moduli space)
3. Coble type hypersurfaces, singular along these loci
4. presumably, uniquely characterized by this property
5. and whose equations we can compute explicitely.

These equations define $W_{\mathfrak{c}}$-equivariant rational maps

$$\mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}^N$$

whose images are nice modular objects (Segre cubic, Göpel variety, etc.)

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$	$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	2	elliptic	$C \times C, \; g = 1$
$\mathbb{C}^4 \otimes \Delta_{10}$	$\mathbb{P}^3$	4	Kummer	K
	$\check{\mathbb{P}}^3$	4	self dual	$K^* \simeq K$
	$\mathbf{Fl_4}$	**2**	Abelian	$Jac(C), g = 2$
	$\mathbb{Q}^8$	4		
	$\mathbf{OG(2, 10)}$	**2**		Fano fourfold
$\wedge^3 \mathbb{C}^9$	$\mathbb{P}^8$	3	Coble cubic	$Jac(C), \; g = 2$
	$\check{\mathbb{P}}^8$	6	dual sextic	$SU_C(3)$
	$G(6, 9)$	2	*Coble quadric*	$SU_C(3, 1)$
$\wedge^4 \mathbb{C}^8$	$\mathbb{P}^7$	4	Coble quartic	$SU_C(2), \; g = 3$
	$\check{\mathbb{P}}^7$	4	self dual	Kum(C)
	$\mathbf{Fl_8}$	**2**		$Jac(C)$
	$G(4, 8)$	2	*Coble quadric*	$SU_C(2, 1)$
Δ_{16}	$\mathbb{Q}^{14}$	4		$SU_C(2)? \; g = 4$
	$\mathbf{OG(2, 16)}$	**2**		$SU_C(2, 1)?$
$S^2 \mathbb{C}^{2g+2}$	$OG(g, 2g + 2)$	2		$Jac(C)$
	$OG(g - 1, 2g + 2)$	2		$SU_C(2, 1)$