On the automorphisms of Mukai varieties

Laurent Manivel and Thomas Dedieu

Institut de Mathématiques de Toulouse

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For Laurent

Hâte-toi
Hâte-toi de transmettre
Ta part de merveilleux de rébellion de bienfaisance
(René Char, Commune Présence)

Laurent's works around θ-representations

Laurent GRUSON, Steven SAM, Jerzy WEYMAN, *Moduli of abelian varieties, Vinberg θ-groups, and free resolutions*, Commutative algebra, 419–469, 2013.

Laurent GRUSON, Steven SAM, *Alternating trilinear forms on a nine-dimensional space and degenerations of* $(3,3)$*-polarized Abelian surfaces*, Proc. Lond. Math. Soc. 110 (2015), 755–785.

A project to be continued...
In April 2019, Laurent was ready to (re)start.

Prime Fano threefolds

Classification of prime Fano threefolds (Fano, Iskhovskih). Smooth projective threefolds X such that $Pic(X) = \mathbb{Z}(-K_X)$ and $-K_X$ ample. When the anticanonical map is an embedding, codimension two linear sections are *canonical curves* of genus g.

g	X	g	X
2	Double sextic	7	section of Spinor variety
3	Quartic in $\mathbb{P}^4$	8	section of Grassmannian
4	Quadric $\cap$ cubic in $\mathbb{P}^5$	9	section of Lagrangian Grass.
5	Three quadrics in $\mathbb{P}^6$	10	section of adjoint variety
6	$G(2,5) \cap Q \cap L$	12	tri-isotropic Grassmannian

Mukai: vector bundle method $\rightsquigarrow$ classification of smooth complex projective manifolds X of dimension $n \geq 4$ such that

$$Pic(X) = \mathbb{Z}H \qquad \text{and} \qquad K_X = -(n-2)H.$$

These **Mukai varieties** are extensions of prime Fano threefolds.
Linear sections of Mukai varieties $\rightsquigarrow$ Mukai varieties.

Conversely, for $g \geq 6$ there are Mukai varieties of maximal dimension.
For $7 \leq g \leq 10$ they are *rational homogeneous spaces*

$$M_g = G/P \hookrightarrow \mathbb{P}(V_g).$$

g	G	V_g	$\dim(V_g)$	M_g	$\dim(M_g)$
7	$Spin_{10}$	Δ_+	16	S_{10}	10
8	SL_6	$\wedge^2\mathbb{C}^6$	15	$G(2,6)$	8
9	Sp_6	$\wedge^{\langle 3\rangle}\mathbb{C}^6$	14	LG(3,6)	6
10	G_2	$\mathfrak{g}_2$	14	$X_{ad}(G_2)$	5

Automorphism groups

Main question for today:

What can be the automorphism group of a Mukai variety??

Focus on genus 7 to 10.
Different behaviors when the codimension increases.

- $X = M_g$, then $Aut(X) = G/Z(G)$.
- Hyperplane section: still a big automorphism group.
- Dimension bigger than critical: positive dimensional group.
- Small dimension: trivial? Not even clear for general prime Fano threefolds.

Theorem (Kuznetsov-Prokhorov-Shramov 2016)

If X is any smooth prime Fano threefold of genus g, with $7 \leq g \leq 10$, the automorphism group $Aut(X)$ is finite.

Hyperplane sections

Start with $M_g = G/P \subset \mathbb{P}(V_g)$.
A hyperplane section is defined by a point in $\mathbb{P}(V_g^\vee)$, which is *quasi-homogeneous*: G acts with an open orbit.

Consequence

Up to isomorphism, $\exists$ unique smooth hyperplane section X of M_g.
$Aut(X)$ is the generic stabilizer of the G-action on $\mathbb{P}(V_g^\vee)$.

How can we lift an automorphism $g \in Aut(X)$ to G?

Mukai: Take more sections to reduce to K3 surfaces of genus g. Then consider the *Mukai bundle* F, a uniquely defined stable vector bundle with special invariants. By unicity, the restrictions of F and g^*F are isomorphic. By cohomological arguments, such an isomorphism lifts to M_g.

CAVEAT! First part is not true in genus $g = 10$, where $V_g = \mathfrak{g}_2$ is the *adjoint representation* of G_2, and M_g is the *adjoint variety*.

$$\begin{array}{ccc} & G_2/B & \\ \swarrow^{\mathbb{P}^1} & & \searrow^{\mathbb{P}^1} \\ \mathbb{Q}^5 & & M_{10} = X_{ad}(G_2) \end{array}$$

In terms of Jordan theory:

- M_g is the projectivization of the minimal nilpotent orbit.
- A general hyperplane section is defined by a regular semisimple element in $\mathfrak{g}_2$.
- $\rightsquigarrow$ the generic stabilizer is essentially a maximal torus.
- $\rightsquigarrow$ one-dimensional family of hyperplane sections.

Adjoint varieties

More generally, there is an adjoint variety $M_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$ (with contact structure, etc.).

Singular hyperplane sections correspond to points on the dual variety $M_{\mathfrak{g}}^{\vee} \subset \mathbb{P}(\mathfrak{g}^{\vee}) = \mathbb{P}(\mathfrak{g})$, a G-invariant hypersurface.

By Chevalley's classical theorem,

$$\mathfrak{g}//G \simeq \mathfrak{t}/W$$

and the equation of the dual is the product of the long roots.

Theorem (Prokhorov-Zaidenberg 2021)

The automorphism group of a smooth hyperplane section of $M_{\mathfrak{g}_2}$ is

$$(G_m^2) \rtimes \mathbb{Z}_2, \quad (G_m^2) \rtimes \mathbb{Z}_6, \quad (G_m \times G_a) \rtimes \mathbb{Z}_2, \quad GL_2 \rtimes \mathbb{Z}_2.$$

Below the critical codimension

Back to to X Mukai variety of genus $7 \leq g \leq 10$.

Naive dimension count $\rightsquigarrow$ there is an integer c_g such that any linear section of M_g of codimension $< c_g$ must have positive dimensional automorphism group:

$$c_7 = 4, \quad c_8 = 3, \quad c_9 = c_{10} = 2.$$

Expectation: the general linear section of codimension $\geq c_g$ should have trivial automorphism group?

Theorem 1 (Dedieu-M.)

The general linear section of M_g of codimension $> c_g$ has trivial automorphism group.

At the critical codimension

Corollary

The general prime Fano threefold of genus $g \geq 7$ has no automorphisms.

$g = 6$ (Gushel-Mukai threefolds): Debarre-Kuznetsov 2018
$g = 12$: Prokhorov 2021 (beware the Mukai-Umemura threefold has infinite automorphism group).

Most interesting case: when the codimension is critical.

Theorem 2 (Dedieu-M.)

The general section of M_g of codim c_g has automorphism group

$$\mathbb{Z}_2^2, \qquad \mathbb{C}^* \rtimes (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2), \qquad \mathbb{Z}_2^4, \qquad 1.$$

General approach

Consider $X = M_g \cap L$. By Mukai's method, one shows that $Stab_G(L) \to Aut(X)$ is surjective. Then we observe that if $u \in Stab_G(L)$, then also $u_s, u_n \in Stab_G(L)$. So we wonder:

Can a non trivial semisimple/unipotent element in G stabilize a general subspace L of V_g of codimension c?

This can be attacked systematically:

- If u is semisimple, it acts on V_g with eigenspaces $E_1, \ldots, E_m$, and the subspaces stabilized by g are parametrized by products of Grassmannians

$$G(\ell_1, E_1) \times \cdots \times G(\ell_m, E_m).$$

- If u is unipotent, similar control on the dimensions of the sets of stable subspaces of each dimension.

Stratifications

Finitely many unipotent orbits to consider, classified.
Similarly, one can stratify the semisimple orbits according to the sizes of their eigenspaces in V_g. In order to prove that the generic automorphism group is trivial in codimension c, enough to prove that for each stratum S,

$$\dim(S) + d_c(S) < \dim G(c, V_g).$$

$\rightsquigarrow$ finite algorithm.

Surprise: there exist a few strata S for which

$$\dim(S) + d_c(S) = \dim G(c, V_g).$$

This happens in codimension $c = c_g$, only for semisimple strata, parametrizing automorphisms of order two for $g \neq 8$.

Genus 8

Recall $M_8 = G(2,6) \subset \mathbb{P}(\wedge^2\mathbb{C}^6)$
$\rightsquigarrow$ Consider $L \subset \wedge^2\mathbb{C}^6$ general of dimension 3 to 12.

Unipotent orbits classified by partitions of 6 $\rightsquigarrow$ cannot stabilize L.

Semisimple elements are determined by eigenvalues $t_1, \ldots, t_6$.
Eigenvalues of the induced action on $\wedge^2\mathbb{C}^6$ are the $t_i t_j$, $i < j$
$\rightsquigarrow$ one has to consider the multiplicities. Beware of:

- Degenerations: some t_i's can coincide;
- Collapsings: $t_i t_j = t_k t_\ell$ with $(i,j) \cap (k,\ell) = \emptyset$.

Laborious but efficient!
Conclusion: L generic of dimension 4 to 11 has no stabilizer.
Thus

$$Aut(X) = 1 \qquad \text{for} \quad X = G(2,6) \cap \mathbb{P}(L^\perp)$$

general Mukai variety of genus 8 and dimension 3, 4.

For $X = G(2,6) \cap \mathbb{P}(L^\perp)$ of dimension 5, so $\dim L = 3$, the conclusion is different.

By the previous analysis L can only be stabilized by:

- $s = z id_A + z^{-1} id_B$ for $\mathbb{C}^6 = A \oplus B$ s.t. $L \subset A \otimes B \subset \wedge^2 \mathbb{C}^6$.
- Some involutions $t = id_E - id_F$ for $\mathbb{C}^6 = E \oplus F$ such that $L = L_1 \oplus L_2$ with $L_1 \subset \wedge^2 E \oplus \wedge^2 F$ and $L_2 \subset E \otimes F$.
- Order 3 elements $u = id_P + j id_Q + j^2 id_R$ with $\mathbb{C}^6 = P \oplus Q \oplus R$, such that L is the sum of three lines contained in the three (five dimensional) eigenspaces of the induced action on $\wedge^2 \mathbb{C}^6$.

$\rightsquigarrow$ How can such transformations fit together?

The first type gives the connected component $\mathbb{C}^* \rightsquigarrow$ The pair (A, B) such that $L \subset A \otimes B$ must be unique (normalization).

Genus 8, continued

Special feature for this case:
The Pfaffian cubic cuts on $\mathbb{P}(L)$ a plane cubic C and $Stab(L)$ has to act on C. How?

- Automorphisms of the first type act trivially.
- Involutions of the second type act as symmetries w. respect to inflexion points of C.
- Order three elements act as translations by 3-torsion points.

Hence the conclusion:

$$Aut(X)/Aut^0(X) \simeq Aut_{lin}(C) \simeq (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2.$$

Genus 7

Here M_7 is the *spinor variety* of dimension 10, index 8 in $\mathbb{P}^{15}$. Small codimensional linear sections were considered before.

- In codimension 1, a unique smooth section; quasi-homogeneous with non reductive automorphism group.
- In codimension 2, two different types of smooth sections. The general one is quasi-homogeneous under $G_2 \times PSL_2$ (Fu-M. 2018). The special one is a compactification of $\mathbb{C}^8$ (Fu-Hwang 2018), with non reductive automorphism group. $\rightsquigarrow$ Counter-examples to rigidity properties for prime Fano manifolds of high index.
- In codimension 3, four different types of smooth sections. Most special one is a compactification of $\mathbb{C}^7$. General one has automorphism group PSL_2^2, not quasi-homogeneous.

Genus 7, codimension four

We focus on codimension 4 $\rightsquigarrow$ Fano sixfolds X of index four, defined by a general $L \subset \Delta$, with Δ the *spin representation*.

The analysis of semisimple/unipotent elements yields:

L may be stabilized by finitely many involutions $s_U = id_U - id_{U^\perp}$ in SO_{10}, where U is some non-degenerate four plane in $\mathbb{C}^{10}$;
$\rightsquigarrow$ splits Δ into two eight dimensional eigenspaces Δ_+ and Δ_-, and $L = L_+ \oplus L_-$ for two planes $L_\pm \subset \Delta_\pm$.

To understand: do these involutions really exist in general? if yes, how many of them? which group do they generate?

The answer to the first question is YES by a dominance argument. The answer to the last two is given by the following Theorem.

Main Theorem for $g = 7$

There exist three non degenerate orthogonal planes A, B, C s.t.

$$Aut(X) = \{1, s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}\} \simeq \mathbb{Z}_2^2.$$

Rough sketch of proof.

$\dim G(4,\Delta) = 4 \times (16 - 4) = 48$. For three orthogonal planes in $\mathbb{C}^{10}$ there are $16 + 12 + 8 = 36$ parameters.

When fixed, the spin representation decomposes into 4 four-dimensional spaces and L must meet each of them along a line, hence 4×3 extra parameters. Since $36 + 12 = 48$ we expect a finite non zero number of triples (A, B, C) for each L.

This is proved to be correct by computing a suitable differential.

Then L is stabilized by the three involutions $s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}$.
If t is another involution stabilizing L, the products $ts_{A\oplus B}, ts_{B\oplus C}, ts_{C\oplus A}$ must be involutions of the same type.
In particular t commutes with $s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}$
$\rightsquigarrow$ more structure for L and contradiction by dimension count.
As a consequence the triple (A, B, C) is unique. □

Remark. For each non trivial involution s in $Aut(X)$, the fixed locus $Fix(s)$ is the union of two quartic surface scrolls in X (codimension two sections of $\mathbb{P}^1 \times \mathbb{P}^3$).
This comes from the exceptional isomorphisms

$$\mathfrak{so}(U) \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2, \qquad \mathfrak{so}(U^\perp) \simeq \mathfrak{sl}_4.$$

Genus 9

Recall M_9 is the Lagrangian Grassmannian $LG(3,6)$, of dimension 6, index 4 in $\mathbb{P}^{13}$. Parametrizes three-planes in $\mathbb{C}^6$ that are isotropic w.t. to a skew-symmetric form ω.
Get Plücker embedding inside $\mathbb{P}(\wedge^{\langle 3\rangle}\mathbb{C}^6)$ where

$$\wedge^{\langle 3\rangle}\mathbb{C}^6 = Ker(\wedge^3\mathbb{C}^6 \xrightarrow{\omega} \mathbb{C}^6).$$

$LG(3,6)$ is a variety with *one apparent double point* (VOADP): a unique bisecant to a general point.

There is a unique smooth hyperplane section of $LG(3,6)$, whose automorphism group is PSL_3.

We focus on codimension two sections, defined by a (co)dimension two subspace $L \subset \wedge^{\langle 3\rangle}\mathbb{C}^6$.

Genus 9, continued

Analysis of semisimple/unipotent elements $\rightsquigarrow$ a generic L can only be stabilized by involutions, of two possible types:

- $s_A = id_A - id_{A^\perp}$ for $A \subset \mathbb{C}^6$ a non isotropic plane,
- $t = id_E - id_F$ for $\mathbb{C}^6 = E \oplus F$ a decomposition into Lagrangian subspaces.

These possibilities can be realized in at most finitely many ways.

How many? For type I, the answer is given by the

Normalization Lemma

There exists a unique triple (A, B, C) of transverse planes in $\mathbb{C}^6$ such that $L \subset A \otimes B \otimes C$.

Genus 9, continued

$\rightsquigarrow$ Chain of VOADP's:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{Q}^3 \subset LG(3,6) \subset G(3,6).$$

Main Theorem for $g = 9$

The stabilizer of a general L is isomorphic with $\mathbb{Z}_2^4$, with

- 3 type I involutions s_A, s_B, s_C,
- 12 type II involutions.

Geometrically, the corresponding involutions in $Aut(X)$ can be distinguished by their fixed locus:

- a del Pezzo surface of degree four in type I,
- the union of two Veronese surfaces in type II.

Relations with θ-representations

QUESTION: Where does all this come from??

Consider the situation where a simple Lie-algebra $\mathfrak{g}$ is endowed with an automorphism θ of finite order p. This yields a $\mathbb{Z}_p$-grading

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{p-1}.$$

Then $(G_0, \mathfrak{g}_1)$ is called a θ-representation.

One can define a Cartan suspace of $(\mathfrak{g}, \theta)$ as a maximal subspace $\mathfrak{h} \subset \mathfrak{g}_1$ of commuting semisimple elements $\rightsquigarrow$ generalized Weyl group $W = N(H)/H$ (a complex reflection group).

Generalized Chevalley theorem

$$\mathfrak{g}_1//G_0 \simeq \mathfrak{h}/W.$$

Examples of θ-representations

For example, we have the following gradings:

$$\mathfrak{f}_4 = \mathfrak{sl}_2 \times \mathfrak{sp}_6 \oplus (\mathbb{C}^2 \otimes \wedge^{\langle 3 \rangle}\mathbb{C}^6),$$
$$\mathfrak{e}_7 = \mathfrak{sl}_3 \times \mathfrak{sl}_6 \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6) \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6)^\vee,$$
$$\mathfrak{e}_8 = \mathfrak{sl}_4 \times \mathfrak{so}_{10} \oplus (\mathbb{C}^4 \otimes \Delta) \oplus (\wedge^2\mathbb{C}^4 \otimes \mathbb{C}^{10}) \oplus (\mathbb{C}^4 \otimes \Delta)^\vee.$$

This means that

- codimension two sections of $LG(3,6)$ are connected to $\mathfrak{f}_4$,
- codimension three sections of $G(2,6)$ are connected to $\mathfrak{e}_7$,
- codimension four sections of Δ are connected to $\mathfrak{e}_8$!

$\rightsquigarrow$ The generic automorphism groups in critical codimension are what remains of the generalized Weyl groups.

Thanks for your attention!