Introduction to Orbital Degeneracy Loci

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Introduction

Motivations

MAIN GOAL: Construct interesting varieties.

- In complex algebraic geometry: classification problems.
- Need to construct varieties of a given class. Typically Fano varieties.
- Very special varieties. For example with particular symmetries.

MAIN INGREDIENTS:

- Vector bundles: very classical.
- Lie theory, representations of algebraic groups: even more classical.

$\rightsquigarrow$ New tools by putting them together.

Basics on vector bundles

Let E be a rank e *vector bundle* over a variety X. So $\pi : E \to X$ is locally trivial and each fiber $E_x = \pi^{-1}(x)$ is identified with a fixed vector space F.

Examples. T_X tangent bundle, rank = dimension of X,
$K_X = \det(\Omega_X) = \det(T_X)^*$ *canonical bundle* (fundamental invariant), rank one i.e. *line bundle*.

A *section* s of E is a map $s : X \to E$ such that $\pi \circ s = id_X$, i.e. $s(x)$ is a point in the vector space E_x. Its *zero-locus* is

$$Z(s) := \{x \in X,\ s(x) = 0 \in E_x\}.$$

Locally over $U \subset X$, decompose $s(x) = s_1(x)f_1 + \cdots + s_e(x)f_e$, for $f_1, \ldots, f_e$ a basis of F. Then $Z = Z(s)$ is defined by the vanishing of the e functions $s_1(x), \ldots, s_e(x)$.

Zero Loci of sections, basic examples

If everywhere transverse, then:

- Z is smooth (possibly empty),
- the codimension of Z is equal to the rank of E,
- the structure sheaf of Z can be resolved by a Koszul complex

$$0 \to \wedge^e E^* \stackrel{s}{\to} \cdots \to E^* \stackrel{s}{\to} \mathcal{O}_X \to \mathcal{O}_Z \to 0,$$

- the canonical bundle of Z is given by the *adjunction formula*

$$K_Z = K_X \otimes \det(E)_{|Z}.$$

Applications.

Global construction of many interesting varieties.
With access to lots of informations on their geometry.

Zero Loci of sections, more examples

Basic examples.

- Zero loci of sections of line bundles (rank one) are hypersurfaces.
- On the Grassmannian $X = G(r, n)$, let T be the tautological vector bundle ($T_x = U$ if x represents $U \subset \mathbb{C}^n$). A linear form ϕ on $\mathbb{C}^n$ defines a section s_ϕ of the dual vector bundle T^*, and

$$Z(s_\phi) = \{U \subset Ker(\phi)\} \simeq G(r, n-1).$$

- $X = G(r, n)$, $E = \wedge^2 T^*$.
  A skew-symmetric two-form ω on $\mathbb{C}^n$ gives a section s_ω, and

$$Z(s_\omega) = IG_\omega(r, n)$$

  is an *isotropic Grassmannian*. For n even, ω of maximal rank, this is again a homogeneous variety, under $Sp(\omega)$.

Zero Loci of sections, more and more examples

More examples

- $X = G(4,7)$, $E = \wedge^3 T^*$.
  A skew-symmetric three-form Ω defines a section s_Ω, and for Ω general $Z = Z(s_\Omega)$ is an eightfold with $K_Z = \mathcal{O}_Z(-4)$.
  This is a Fano variety, quasi-homogeneous under G_2, actually a smooth completion of the symmetric space $G_2/SO_3 \times SO_3$.

- $X = G(3,7)$, $E = \wedge^2 T^* \oplus \wedge^2 T^* \oplus \wedge^2 T^*$.
  A triple $\theta = (\theta_1, \theta_2, \theta_3)$ of skew-symmetric two-forms defines a section s_θ, and

  $$Z = Z(s_\theta) = IG_{\theta_1}(3,7) \cap IG_{\theta_2}(3,7) \cap IG_{\theta_3}(3,7)$$

  is a threefold with $K_Z = \mathcal{O}_Z(-1)$: a prime Fano threefold.
  Actually a whole family of such threefolds.

Zero Loci of sections

Still more examples

- $X = G(2,6)$, $E = S^2T^* \oplus S^2T^*$.
  A pair of quadrics $q = (q_1, q_2)$ in 6 variables defines a section s_q, and $Z = Z(s_q)$ is a surface parametrizing projective lines contained in $Q_1 \cap Q_2 \subset \mathbb{P}^5$.
  $\rightsquigarrow$ $K_Z = \mathcal{O}_Z$: Z is an Abelian surface (Reid 1972).

- $X = G(2,6)$, $E = S^3T^*$.
  A degree three polynomial P in 6 variables defines a section s_P, and $Z = Z(s_P)$ is a fourfold parametrizing projective lines in the cubic hypersurface $X(P) \subset \mathbb{P}^5$.
  $\rightsquigarrow$ $K_Z = \mathcal{O}_Z$: Z is hyperKähler (Beauville-Donagi 1985).

Zero Loci of sections, constraints

More restricted goals

- Construct Fano manifolds, $K_Z < 0$ (finite problem).
- Construct Calabi-Yau and hyperKähler manifolds, $K_Z = 0$.

Constraints: double bind!

- If E has enough sections, then $\det(E) > 0$.
- We want $K_Z = K_X \otimes \det(E)_{|Z} \leq 0$, so need $K_X < 0$.
- So we need *low rank vector* bundles on Fano manifolds, *positive but not too much!*
- Classical conjecture: low rank vector bundles on projective spaces are split (sums of lines bundles).

$\rightsquigarrow$ Need more flexibility!

Determinantal Loci

Suppose $E = Hom(F, G)$ for two vector bundles F, G on X.
A section of E is a morphism $\varphi : F \to G$, with variable rank.
So one defines the k-th *determinantal locus*

$$D_k(\varphi) := \{x \in X,\ \mathrm{rank}(\varphi_x) \leq k\}.$$

For φ general, $D_k = D_k(\varphi)$ is no longer smooth, but

- the codimension of D_k is $(f-k)(g-k)$,
- the singular locus of D_k is

$$\mathrm{S}ing(D_k) = D_{k-1},$$

- if $(f-k)(g-k) < \dim(X) < (f-k+1)(g-k+1)$, then $D_{k-1} = \emptyset$, D_k is smooth, and ϕ has constant rank on D_k.

Determinantal Loci, continued

So there is an exact sequence

$$0 \to A \to F \stackrel{\varphi}{\to} G \to B \to 0,$$

on D_k, with $A = \mathrm{K}er(\varphi)$ and $B = \mathrm{C}oker(\varphi)$. Then the normal bundle of D_k is $N = Hom(A, B)$, and

$$K_{D_k} = K_{X|D_k} \otimes \det(N).$$

But K_{D_k} difficult to control: not a restriction!

Exception: $f = g$, since then

$$K_{D_k} = K_X \otimes \det(F^*)^{g-k} \otimes \det(G)^{f-k}_{|D_k}.$$

Advantage. Can construct interesting loci (CY threefolds) just from line bundles or very simple vector bundles.

Principal bundles

For determinantal loci, we have used the fact that $E = Hom(F, G)$ has extra structure $\rightsquigarrow$ the rank is the invariant that describes in which stratum the section lands at any given point.

More generally: suppose that E has extra structure, encoded by some representation V of some complex Lie group G.

In technical language, we need a G-principal bundle $P \to X$, and $E = E_V$ is the associated bundle to P and the representation V. Then each fiber of E can be identified with V, not canonically but only up to the action of G.

Typical example: $G = GL(f, \mathbb{C})$ and $V = \wedge^k \mathbb{C}^f$. Then this is equivalent to asking that there exists a vector bundle F of rank f on X, such that $E \simeq \wedge^k F$.

Orbital Degeneracy Loci, definition

For a section s of E, the stratum in which s lands at a given point $x \in X$ is the G-orbit in V to which s_x belongs.
$\rightsquigarrow$ Suggests to define, for $Y \subset V$ any closed G-invariant subset,

$$D_Y(s) := \{x \in X,\ s_x \in Y \subset V \simeq E_x\}.$$

Typically, Y will be an orbit closure. In this case we get an *orbital degeneracy locus* (ODL).

Facts. For s general (transverse),

- the codimension of $D_Y(s) \subset X$ equals that of $Y \subset V$,
- the singular locus of $D_Y(s)$ is

$$\mathrm{Sing}(D_Y(s)) = D_{\mathrm{Sing}(Y)}(s).$$

Orbital Degeneracy Loci, problems

Several problems to deal with.

Problem A. Understand G-orbits in a G-representation V.

$\rightsquigarrow$ Hopeless in general! We will first restrict to *parabolic representations*, which have *finitely many* orbits.

Problem B. Describe the geometry of the G-orbit closures.

$\rightsquigarrow$ In particular, describe the singularities. Or rather, resolve the singularities.

Problem C. Construct varieties Z with $K_Z \leq 0$ as ODL, or resolutions of ODL.

$\rightsquigarrow$ As before, we will need to start from low rank vector bundles (possibly just line bundles) on Fano varieties.

Classification of orbits

Problem A (classification of orbits) is very classical.

G simple complex Lie group $\rightsquigarrow$ complete classification of representations with finitely many G-orbits (Kac 1981).
Most of them are *parabolic* ($\simeq$ gradings of simple Lie algebras).
Moreover, orbits have been described explicitly by various means (e.g., by normal forms).

Examples

1. $G = Gl(f,\mathbb{C})$, $V = \mathbb{C}^f$: two orbits $\rightsquigarrow$ zero loci of sections.
2. $G = Gl(f,\mathbb{C}) \times Gl(g,\mathbb{C})$, $V = Hom(\mathbb{C}^f, \mathbb{C}^g)$: orbits defined by the rank $\rightsquigarrow$ determinantal loci of morphisms.
3. $G = Gl(f,\mathbb{C})$, $V = \wedge^2\mathbb{C}^f$: rank $\rightsquigarrow$ Pfaffian loci.
4. $G = Gl(f,\mathbb{C})$, $V = \wedge^3\mathbb{C}^f$: finitely many orbits only for $f \leq 8$.

Descriptions of orbits

Problem B (geometry of orbits) was partially solved by Weyman & al, in terms of *Kempf collapsings*.

Method. Let $P \subset G$ be a parabolic subgroup and let $U \subset V$ be a P-stable subspace. The G-translates of U form a vector bundle $\mathcal{U}$ over the flag manifold G/P.

$$\begin{array}{ccc} & \mathcal{U} & \\ {}^{p}\swarrow & & \searrow^{q} \\ G/P & & Y \subset V. \end{array}$$

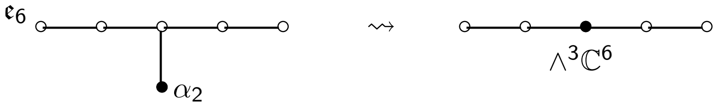
The map q is proper, so $Y = q(\mathcal{U})$ is closed and G-stable $\rightsquigarrow$ orbit closure. If q is birational, it is a resolution of singularities.

A toy example, 1

3-forms in 6 variables.

Start with $\mathfrak{e}_6$, the 78-dimensional exceptional simple complex Lie algebra. The simple root α_2 defines a $\mathbb{Z}$-grading

$$\mathfrak{e}_6 = \mathbb{C} \oplus \wedge^3\mathbb{C}^6 \oplus \mathfrak{gl}_6 \oplus \wedge^3\mathbb{C}^6 \oplus \mathbb{C}.$$



Orbits of $GL(6,\mathbb{C})$ in $\wedge^3\mathbb{C}^6$ are traces of nilpotent orbits in $\mathfrak{e}_6$
$\rightsquigarrow$ finiteness!

A toy example, 2

3-forms in 6 variables, continued

Let $G = GL(6, \mathbb{C})$ act on $V = \wedge^3 \mathbb{C}^6$. The orbit closures are

$$Y_{20} = \{0\} \subset Y_{10} \subset Y_5 \subset Y_1 \subset Y_0 = V.$$

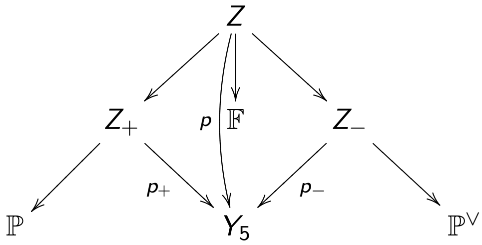
The orbits have the following representatives ($e_{ijk} = e_i \wedge e_j \wedge e_k$):

$\mathcal{O}_{10}$	e_{123}	decomposable
$\mathcal{O}_5$	$e_{123} + e_{145}$	partially decomposable
$\mathcal{O}_1$	$e_{124} + e_{135} + e_{236}$	tangent or dual quartic
$\mathcal{O}_0$	$e_{123} + e_{456}$	generic

Taking closure: $Y_5 = \bar{\mathcal{O}}_5 = \mathcal{O}_5 \cup \mathcal{O}_{10} \cup \mathcal{O}_{20} = \mathcal{O}_5 \cup Y_{10}$.

A toy example, 3

The variety Y_5 is singular along Y_{10}. One can resolve the singularities by three Kempf collapsings p, p_+, p_-.



Here $\mathbb{P} = \mathbb{P}(\mathbb{C}^6)$, $\mathbb{P}^\vee = \mathbb{P}(\mathbb{C}^6)^*$ and $\mathbb{F} \subset \mathbb{P} \times \mathbb{P}^\vee$ is the flag variety parametrizing pairs (line $\subset$ hyperplane).

A toy example, 4

The desingularizations Z, Z_+, Z_- are total spaces of homogeneous vector bundles $\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-$, where

$$\mathcal{E}_+(U_1) = U_1 \wedge (\wedge^2 \mathbb{C}^6), \qquad \mathcal{E}_-(U_5) = \wedge^3 U_5,$$

$$\mathcal{E}(U_1 \subset U_5) = U_1 \wedge (\wedge^2 U_5).$$

Observations.

1. $p_\pm^{-1}(e_{123}) \simeq \mathbb{P}^2$, so $p_\pm$ is a *small contraction*,
2. $p^{-1}(e_{123}) \simeq \mathbb{P}^2 \times \mathbb{P}^2$, so p is a *divisorial contraction*,
3. the determinant of $\mathcal{E}_-$ is

$$\det(\mathcal{E}_-) = \wedge^{10}(\wedge^3 U_5) = \det(U_5)^6 = K_{\mathbb{P}^\vee},$$

and therefore p_- (and p_+ as well) is a *crepant resolution*.

Resolving the structure sheaf

By pushing forward Koszul complexes from Z, Weyman & al showed that the structure sheaf of Y_5 admits a very beautiful self-dual resolution

$$0 \to \mathcal{O}_V(-10) \to V \otimes \mathcal{O}_V(-7) \to \mathfrak{sl}_6 \otimes \mathcal{O}_V(-6) \to \cdots$$
$$\cdots \to \mathfrak{sl}_6 \otimes \mathcal{O}_V(-4) \to V \otimes \mathcal{O}_V(-3) \to \mathcal{O}_V \to \mathcal{O}_{Y_5} \to 0.$$

Allows to compute the Hilbert function, etc.
$\rightsquigarrow$ very complete understanding, useful for applications to Problem C.

Nilpotent orbits

Many parabolic representations and many orbits in each of them $\rightsquigarrow$ rich playground.
Only a small proportion have crepant resolutions.

Representations with infinitely many orbits are also interesting.
Typically:

$$\begin{array}{ccc} & \Omega_{G/P} & \\ {}^{p}\swarrow & & \searrow^{q} \\ G/P & & \bar{\mathcal{O}}_P \subset \mathfrak{g}. \end{array}$$

Nilpotent orbits $\mathcal{O}_P$ obtained from parabolic subgroups P of G are called Richardson orbits. If q is birational, it is automatically crepant since $K_{G/P} = \det(\Omega_{G/P})$!

Application: constructing Fano or CY varieties

Problem C can now be solved.

Suppose we have a rank 6 vector bundle F on a variety X, and a generic section s of $E = \wedge^3 F$. The ODL $D = D_{Y_5}(s)$ is the locus of points $x \in X$ where $s_x \in \wedge^3 F_x$ becomes partially decomposable.

- D has codimension 5, and is singular in codimension 5,
- D admits explicit resolutions of singularities,
- D is Gorenstein with canonical bundle

$$K_D = K_X \otimes (\det F)^5_{|D}.$$

For example we can construct (a dozen of) Calabi-Yau fourfolds from (X, F), $\dim(X) = 9$, $\mathrm{rank}(F) = 6$, $K_X \otimes (\det F)^5$ trivial.

3-forms in 9 variables, 1

The action of $G = GL(9, \mathbb{C})$ on $V = \wedge^3\mathbb{C}^9$ has infinitely many orbits ($81 < 84$). Can we understand them geometrically?

Reduction. Can contract $\Omega \in \wedge^3\mathbb{C}^9$ by a linear form to get a skew-symmetric 2-form. Define

$$H_\Omega := \{P \in \mathbb{P}^\vee, \mathrm{rank}\, \Omega(P, \bullet, \bullet) \leq 6\},$$

$$A_\Omega := \{P \in \mathbb{P}^\vee, \mathrm{rank}\, \Omega(P, \bullet, \bullet) \leq 4\}.$$

These are Pfaffian loci in $\mathbb{P}^\vee = \mathbb{P}(\mathbb{C}^9)^* \simeq \mathbb{P}^8$.

Theorem (Gruson-Sam-Weyman 2013)

For Ω generic, H_Ω is a cubic hypersurface, with singular locus A_Ω, a smooth **abelian surface**.

3-forms in 9, 2: Coble cubics

$\rightsquigarrow$ One recovers a classical situation!

Definition. An abelian surface $A = \mathbb{C}^2/\Lambda$ is *principally polarized* if endowed with an ample line bundle L such that $h^0(A, L) = 1$.

Then $h^0(A, L^{\otimes n}) = n^2$ and

1. the sections of $L^{\otimes 2}$ define a degree 2 morphism $A \to H \subset \mathbb{P}^3$, with H a singular quartic surface,
2. the sections of $L^{\otimes 3}$ define an embedding $A \hookrightarrow \mathbb{P}^8$. Coble (1918): *$A$ is the singular locus of a unique cubic hypersurface.*
3. A is the Jacobian of a genus two curve C, and the Coble cubic can be interpreted in terms of moduli spaces of vector bundles on C (Narasimhan-Ramanan 1984, Laszlo 1996, Beauville 2003, Ortega 2005, Dolgachev-Minh 2007).

3-forms in 9 variables, 3: through the looking-glass

Dual reduction. Suppose $U_1 \subset \mathbb{C}^9$ is a line. We can mod out $\wedge^3\mathbb{C}^9$ by U_1 to get $\wedge^3(\mathbb{C}^9/U_1) \simeq \wedge^3\mathbb{C}^8$. Here $GL(8,\mathbb{C})$ has finitely many orbits (22), starting from:

$$\begin{array}{ccccccccccc} & & & & & & & \nearrow & & & Y_8' \cdots \\ Y_0 = \wedge^3\mathbb{C}^8 & \longrightarrow & Y_1 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_6 & \longrightarrow & Y_8 \cdots \end{array}$$

We can thus define the ODL associated to $\Omega \in \wedge^3\mathbb{C}^9$:

$$D_k := D_{Y_k}(\Omega) = \{U_1 \subset \mathbb{C}^9,\ \Omega \bmod U_1 \in Y_k\} \subset \mathbb{P} = \mathbb{P}^8.$$

For Ω generic the locus D_k has codimension k.

3-forms in 9 variables, 4

Facts

1. $D_8 = 81$ points in $\mathbb{P}^8$, while $D_8' = \emptyset$.
2. D_6 is a non normal surface, singular along D_8.
3. D_4 is singular along D_6 and has **trivial canonical bundle**.
4. D_1 is a sextic hypersurface, dual to the Coble cubic.

Suggests to focus on D_4 and its desingularization, deduced from a Kempf collapsing of Y_4:

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ \mathbb{F}_{2,5}(\mathbb{C}^8) & & & & Y_4 \subset \wedge^3\mathbb{C}^8, \end{array}$$

where $\mathcal{E}(U_2 \subset U_5 \subset \mathbb{C}^8) = \wedge^3 U_5 + U_2 \wedge U_5 \wedge \mathbb{C}^8 \subset \wedge^3\mathbb{C}^8$.

3-forms in 9 variables, 5

This induces a desingularization

$$\mathbb{F}_{1,3,6}(\mathbb{C}^9) \xleftarrow{\;i\;} X_4 \xrightarrow{\;\pi\;} D_4 \subset \mathbb{P}.$$

Theorem

X_4 is a hyperKähler fourfold. More precisely, $X_4 \simeq Kum^2(A_\Omega)$, the generalized Kummer fourfold associated to the abelian surface A_Ω.

Beauville: $Hilb^n(A)$ is a smooth manifold of dimension $2n$, and the addition law on A induces a fibration $Hilb^n(A) \to A$, such that every fiber $Kum^{n-1}(A)$ is a hyperKähler manifold.

3-forms in 9 variables, 6

Why do we find that exciting? Because:

- HyperKähler manifolds are rare!
  Only two families known in dimension four, but not known if there exist finitely or infinitely many families.
- Very few projective models of generalized Kummer fourfolds had been described before.
- We can deduce a nice geometric description of the addition law on A_Ω.

Main observation

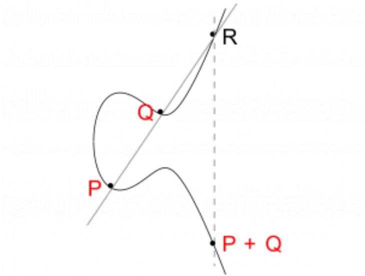
Three general points $P, Q, R \in A_\Omega \subset \mathbb{P}^\vee$ are such that $P + Q + R = O$, a fixed origin in A_Ω, if and only if

$$\Omega(P, Q, \bullet) = \Omega(P, R, \bullet) = \Omega(Q, R, \bullet)$$

give the same point in $\mathbb{P}$.

3-forms in 9 variables, conclusion

Starting from two points P, Q on A_Ω, we thus find $R \in A_\Omega$ as above by solving a problem in linear algebra. Then we do the same with O, R to find the point $S = P + Q$.



Consequences: $D_8 =$ 3-torsion points in A_Ω; $D_6 \subset Hilb^3(A_\Omega)$ is made of schemes with non reduced support, normalization is A_Ω.

References.

Orbital degeneracy loci and applications, with V. Benedetti, S.A. Filippini, F. Tanturri, 36 pages, arXiv:1704.01436, to appear in Annali Scuola Normale di Pisa.

Orbital degeneracy loci II : Gorenstein orbits, with V. Benedetti, S.A. Filippini, F. Tanturri, 40 pages, arXiv:1802.08430, to appear in IMRN.

The geometry of the Coble cubic and orbital degeneracy loci, with V. Benedetti and F. Tanturri, 24 pages, arXiv:1904.10848, to appear in Math. Annalen.

Thank You!