

Prehomogenous spaces and projective geometry

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CHAPTER 1

**Complex semisimple Lie algebras and their
representations**

1.1. Simple Lie algebras

In this lecture we briefly cover some standard material, the study and the classification of complex (semi)simple Lie algebras via their root systems. The reader is encouraged to consult [12, 5, 30, 11] for more details.

1.1.1. The Cartan-Killing classification.

DEFINITION 1.1.1. A Lie algebra is a vector space $\mathfrak{g}$ over a field k , endowed with a skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called *the Lie bracket*, such that

$$\forall X, Y, Z \in \mathfrak{g}, \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

This identity is called *the Jacobi identity*.

One can try to understand finite groups by looking for normal subgroups, or commutative algebras by considering ideals. The same idea can be applied to Lie algebras.

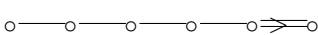
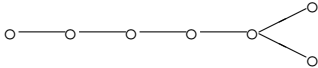
DEFINITION 1.1.2. An *ideal* $\mathfrak{i}$ in a Lie algebra $\mathfrak{g}$ is a subvector space such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$. In particular it is a Lie subalgebra, and the quotient space $\mathfrak{g}/\mathfrak{i}$ has an induced structure of Lie algebra.

If a Lie algebra has a proper ideal (proper means different from zero or the Lie algebra itself), it is built from two smaller Lie algebras.

DEFINITION 1.1.3. A *simple* Lie algebra is a noncommutative Lie algebra without any proper ideal.

Simple complex Lie algebras were classified at the end of the 19th century by W. Killing and E. Cartan. There are four (or rather three) infinite series $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$ of *classical Lie algebras* and five *exceptional Lie algebras* $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

Every simple complex Lie algebra is encoded in a *Dynkin diagram*. The Dynkin diagrams of the classical Lie algebras are as follows (the integer n always denotes the number of vertices; the right-most column gives the dimension):

A_n		$\mathfrak{sl}_{n+1}$	$n^2 + 2n$
B_n		$\mathfrak{so}_{2n+1}$	$2n^2 + n$
C_n		$\mathfrak{sp}_{2n}$	$2n^2 + n$
D_n		$\mathfrak{so}_{2n}$	$2n^2 - n$

These algebras are familiar:

- $\mathfrak{sl}_n$ is the Lie algebra of traceless matrices of size n , the Lie bracket being given by the usual commutator $[X, Y] = XY - YX$;
- the n -th orthogonal Lie algebra $\mathfrak{so}_n \subset \mathfrak{sl}_n$ is the subalgebra of skew-symmetric matrices;

- the n -th symplectic Lie algebra $\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$ is the subalgebra of matrices X such that ${}^tXJ + JX = 0$, where J is any invertible skew-symmetric matrix; up to isomorphism, this algebra does not depend on J .

The Dynkin diagrams of the exceptional Lie algebras are:

G_2		$\mathfrak{g}_2$	14
F_4		$\mathfrak{f}_4$	52
E_6		$\mathfrak{e}_6$	78
E_7		$\mathfrak{e}_7$	133
E_8		$\mathfrak{e}_8$	248

1.1.2. The Killing form. On any Lie algebra $\mathfrak{g}$, one can define a symmetric bilinear form as follows. For any $x \in \mathfrak{g}$, denote by $ad(x)$ the operator on $\mathfrak{g}$ defined by $ad(x)(y) = [x, y]$, and let

$$K(x, y) = \text{trace}(ad(x)ad(y)), \quad x, y \in \mathfrak{g}.$$

This is the Killing form. Despite its name, its importance was stressed by E. Cartan. Its first relevant property is its *invariance*:

$$K(x, [y, z]) = K([x, y], z), \quad x, y, z \in \mathfrak{g}.$$

The Killing form is able to detect important properties of the Lie algebra. For example, a Lie algebra is said to be *semi-simple* if it can be decomposed as a direct sum of simple Lie algebras.

Proposition 1.1.4. (Cartan's criterion) *A Lie algebra $\mathfrak{g}$ is semi-simple if and only if its Killing form is nondegenerate.*

Cartan subalgebras. The main idea to classify simple complex Lie algebras is to diagonalize them with the help of a maximal abelian subalgebra whose elements are all semisimple. By $x \in \mathfrak{g}$ semi-simple we mean precisely that the operator $ad(x)$, acting on $\mathfrak{g}$, is diagonalizable. In the sequel $\mathfrak{g}$ will be simple, and $\mathfrak{t}$ will denote such a maximal subalgebra, which is called a *Cartan subalgebra*.

EXAMPLE 1.1.5. Check that $\mathfrak{sl}_n$ is simple and that the subalgebra of diagonal matrices is a Cartan subalgebra.

Once we have a Cartan subalgebra $\mathfrak{t}$, we can simultaneously diagonalize its adjoint action on $\mathfrak{t}$. We get a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g}, ad(h)(x) = \alpha(h)x \ \forall h \in \mathfrak{t}\}.$$

The Jacobi identity implies that

$$(1.1.1) \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

An easy consequence is that

$$(1.1.2) \quad K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \text{ whenever } \alpha + \beta \neq 0.$$

Since the Killing form is nondegenerate, this allows one to identify $\mathfrak{g}_{-\alpha}$ with the dual of $\mathfrak{g}_\alpha$. In particular, the restriction of the Killing form is nondegenerate on $\mathfrak{g}_0 = C(\mathfrak{t}) = \{x \in \mathfrak{g}, [x, \mathfrak{t}] = 0\}$, the *centralizer* of $\mathfrak{t}$. One can prove that a Cartan subalgebra is its own centralizer, i.e., $\mathfrak{g}_0 = C(\mathfrak{t}) = \mathfrak{t}$.

We can therefore write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where Φ is the set of nonzero linear forms $\alpha \in \mathfrak{t}^*$ such that $\mathfrak{g}_\alpha \neq 0$. These are called the *roots* of the pair $(\mathfrak{g}, \mathfrak{t})$, and $\Phi \subset \mathfrak{t}^*$ is the *root system*.

Note that the Killing form of $\mathfrak{g}$ is nondegenerate on $\mathfrak{t}$, thus allows us to identify $\mathfrak{t}^*$ with $\mathfrak{t}$. In particular, for each root $\alpha \in \mathfrak{t}^*$ we have an element $h_\alpha \in \mathfrak{t}$ defined by $\beta(h_\alpha) = K(\alpha, \beta)$ for all $\alpha, \beta \in \Phi$, where K is dual to the restriction of the Killing form on $\mathfrak{t}$.

1.1.3. Properties of the root system. . The key of the classification of simple Lie algebras will be to show that the *root system* Φ has very special properties.

Proposition 1.1.6. *Φ generates $\mathfrak{t}^*$ and is symmetric with respect to the origin. Moreover, for all $\alpha \in \Phi$,*

- (1) *if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = K(x, y)h_\alpha$,*
- (2) *$\alpha(h_\alpha) \neq 0$,*
- (3) *$\dim \mathfrak{g}_\alpha = 1$.*

This last property, and the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, already give very precise information about the Lie algebra $\mathfrak{g}$. Indeed, we can choose for each root α a generator x_α of $\mathfrak{g}_\alpha$, and the Lie bracket is then given by

$$\begin{aligned} [h, h'] &= 0 && \text{for } h, h' \in \mathfrak{t}, \\ [h, x_\alpha] &= \alpha(h)x_\alpha && \text{for } h \in \mathfrak{t}, \\ [x_\alpha, x_{-\alpha}] &= K(x_\alpha, x_{-\alpha})h_\alpha, \\ [x_\alpha, x_\beta] &= C_{\alpha, \beta}x_{\alpha+\beta}, && \text{when } \alpha + \beta \neq 0, \end{aligned}$$

for some *structure constants* $C_{\alpha, \beta}$ to be determined. To classify the simple Lie algebras, it is therefore necessary to classify the root systems first. To do this we need a deeper understanding of their properties.

The key point is the following observation. Choose a root α , and then $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $K(X_\alpha, X_{-\alpha}) = 2/\alpha(h_\alpha)$. If $H_\alpha = [X_\alpha, X_{-\alpha}]$, we get that $\alpha(H_\alpha) = 2$, and H_α is called the *coroot* of α . We deduce the relations $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, X_{-\alpha}] = -2X_{-\alpha}$, which are precisely the commutation relations of the generators of $\mathfrak{sl}_2$,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This means that $\mathfrak{g}$ contains a copy of $\mathfrak{g}^{(\alpha)} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2$ as a subalgebra. Moreover, for each root β , the space

$$\mathfrak{g}_\beta^{(\alpha)} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$$

is preserved by the adjoint action of $\mathfrak{g}^{(\alpha)}$, and inherits the structure of a $\mathfrak{sl}_2$ -module.

It is easy to construct $\mathfrak{sl}_2$ -modules of arbitrary dimensions. For any integer $k \geq 0$, denote by V_k the space of homogeneous polynomials of degree k in two variables u and v . We can define an action of $\mathfrak{sl}_2$ on V_k by letting the generators X, Y, H of $\mathfrak{sl}_2$ act as

$$X = v \frac{\partial}{\partial u}, \quad Y = u \frac{\partial}{\partial v}, \quad H = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}.$$

One can check that V_k is an *irreducible* $\mathfrak{sl}_2$ -module, which means that it does not contain any proper submodule. Note that H acts on $u^i v^{k-i}$ by multiplication by $2i - k$, and these integers form a chain from $-k, -k+2, \dots$ to $\dots, k-2, k$.

Theorem 1.1.7. *Every finite dimensional $\mathfrak{sl}_2$ -module is completely decomposable into a direct sum of irreducible modules. Every irreducible module of dimension $k+1$ is isomorphic to V_k .*

The information we have on $\mathfrak{sl}_2$ -modules can now be applied to the action of $\mathfrak{g}^{(\alpha)}$ on $\mathfrak{g}_\beta^{(\alpha)}$. Since H_α acts on $\mathfrak{g}_{\beta+k\alpha}$ by multiplication by $\beta(H_\alpha) + 2k$, the roots of $\mathfrak{g}$ of the form $\beta+k\alpha$ must form a chain $\beta-q\alpha, \dots, \beta+p\alpha$ with $\beta(H_\alpha) - 2q = -\beta(H_\alpha) - 2p$, that is $\beta(H_\alpha) = q - p \in \mathbb{Z}$. Consequences:

- (1) for any roots $\alpha, \beta \in \Phi$, $s_\alpha(\beta) := \beta - \beta(H_\alpha)\alpha$ is again a root, since $-q \leq \beta(H_\alpha) = p - q \leq p$;
- (2) if two roots are colinear, they are equal or opposite: indeed, if $\beta = t\alpha$, say $t > 1$, then $2 = \beta(H_\beta) = t\alpha(H_\beta)$, and since $\alpha(H_\beta)$ is an integer this forces $t = 2$, which is impossible.

Note also that for $h, h' \in \mathfrak{t}$, $K(h, h') = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h')$, so that the Killing form is positive definite on a real form of $\mathfrak{t}$. These properties will be the basis of the definition of *abstract reduced root systems*.

1.1.4. Abstract root systems.

DEFINITION 1.1.8. An *abstract root system* is a finite subset Φ of some finite dimension Euclidian (real) vector space V , such that

- (1) $0 \notin \Phi$ and Φ generates V ,
- (2) $\forall \alpha, \beta \in \Phi$, $c_{\beta\alpha} = 2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$,
- (3) $\forall \alpha, \beta \in \Phi$, $s_\alpha(\beta) = \beta - c_{\beta\alpha}\alpha \in \Phi$,
- (4) if $\alpha, \beta \in \Phi$ are colinear, they are equal or opposite.

Note that the definition of s_α extends to V , and that the resulting map is just the orthogonal symmetry with respect to the hyperplane $\alpha^\perp$. In particular, $s_\alpha(\alpha) = -\alpha \in \Phi$.

The subgroup of the orthogonal group of V , generated by the *reflections* s_α , $\alpha \in \Phi$, is a subgroup of the permutation group of Φ . In particular, it is a finite group, called the *Weyl group* of the root system.

Of course, the root system of a simple complex Lie algebra will be an abstract root system. We have already proved it, except that the root system is naturally

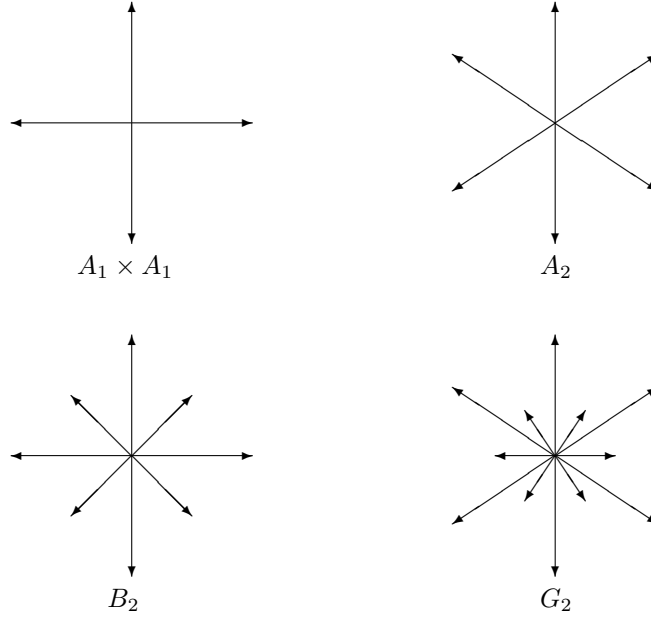
embedded in a complex vector space (the dual $\mathfrak{t}^*$ of a Cartan subalgebra), not in a real Euclidian space. But the real span of the root system is a real subspace of $\mathfrak{t}^*$ whose real dimension is the complex dimension of $\mathfrak{t}$, and the restriction of the Killing form provides it with a Euclidian structure.

- REMARK 1.1.9. (1) The intersection of an abstract root system with any subspace of V is again an abstract root system in the subspace that it generates.
- (2) If we let $\check{\alpha} = \alpha / \langle \alpha, \alpha \rangle$, we get another abstract root system in V , with the same Weyl group. This is called the *dual root system* of Φ .

The dimension of V is called the *rank* of Φ . Up to scale, there is only one abstract root system of rank one, denoted A_1 . In rank two, observe that if α, β are not parallel,

$$c_{\alpha\beta}c_{\beta\alpha} = 4\cos^2(\alpha, \beta) \in \{0, 1, 2, 3\}.$$

This implies that $c_{\alpha\beta}$ or $c_{\beta\alpha}$ is equal to ± 1 . Changing α in β or $-\beta$ if necessary, we can suppose that $c_{\alpha\beta} = -1$. Then for each possible value of $c_{\beta\alpha} \in \{0, -1, -2, -3\}$, the angle between α and β , and (if $c_{\beta\alpha} \neq 0$) their relative lengths are fixed. This being true for any pair of roots, it is a simple exercise to check that any abstract root system of rank two is equivalent, up to isometry, to one of the following ones:



The abstract root system $A_1 \times A_1$ is *decomposable*, i.e., it is a product of two proper systems. The other rank two systems are *indecomposable*. This is equivalent to the fact that the action of the Weyl group on the ambient space is irreducible.

In order to classify abstract root systems of higher rank, let v be a linear form on V that does not vanish on Φ . This splits Φ into the disjoint union of $\Phi^+ = \{\alpha \in \Phi, v(\alpha) > 0\}$ and $\Phi^- = -\Phi^+$. Define a *positive root* $\alpha \in \Phi^+$ to be *simple*, if it cannot be written as the sum of two positive roots.

Proposition 1.1.10. *The set of simple roots $\Delta \subset \Phi^+$ is a basis of V . Moreover, each positive root is a linear combination of simple roots with non negative integer coefficients.*

Now that we have a set of simple roots, we can define the *Dynkin diagram* $D(\Phi)$ as follows. This is an abstract graph whose vertices are in correspondence with the simple roots. If $\alpha, \beta \in \Delta$, we join the corresponding vertices by $c_{\alpha\beta}c_{\beta\alpha} \in \{0, 1, 2, 3\}$ edges; if α and β do not have the same length, these edges are given arrows pointing to the smallest of the two simple roots.

Note that if $D(\Phi)$ is given, the angles and the respective lengths of the simple roots are known, hence the set of simple roots is determined up to isometry. Moreover, the integers $c_{\alpha\beta}$ can also be read off $D(\Phi)$, and this allows one to recover the full Weyl group. Applying the Weyl group to the simple roots, we get the whole root system back (one can show that the Weyl group acts transitively on the set of roots of the same length in an indecomposable root system, and that there can be at most two different lengths: when the roots do not have all the same length, we distinguish between *short roots* and *long roots*). Conclusion: up to isometry, an abstract root system is completely determined by its Dynkin diagram.

EXAMPLE 1.1.11. Consider the Lie algebra $\mathfrak{sl}_n$, and the Cartan subalgebra $\mathfrak{t}$ of diagonal traceless matrices. Let $\varepsilon_i(h) = h_i$, the i -th diagonal coefficient of $h \in \mathfrak{t}$. The adjoint action of $\mathfrak{t}$ on $\mathfrak{sl}_n$ is diagonal in the basis given by a basis of $\mathfrak{t}$ and the set of matrices e_{ij} with only one nonzero coefficient, a one at the intersection of the i -th line and the j -th column, where $i \neq j$. The corresponding root is $\varepsilon_i - \varepsilon_j$, and the Killing form is therefore given on $\mathfrak{t}$ by

$$K(h, h) = \sum_{i < j} (h_i - h_j)^2 = n \sum_i h_i^2.$$

Choose the positive roots to be the $\varepsilon_i - \varepsilon_j$ with $i < j$. The simple roots are then the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i < n$, and we get that $c_{\alpha_i \alpha_j} = -1$ if $|i - j| = 1$ and 0 otherwise. The Dynkin diagram is thus a chain of length $n - 1$, denoted by A_{n-1} .

The matrix $(c_{\alpha\beta})_{\alpha, \beta \in \Delta}$ is called the *Cartan matrix*. It is an integer matrix, whose diagonal coefficients are equal to two and all other coefficients are nonpositive. Moreover, it has the important property that the closely related matrix $(\langle \beta, \beta \rangle c_{\alpha\beta})_{\alpha, \beta \in \Delta}$, is symmetric and positive definite, since it is just the matrix of the Killing form (or rather its dual) in the basis of simple roots.

EXAMPLE 1.1.12. Consider the matrix

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

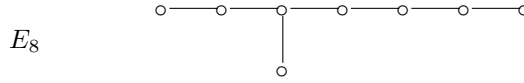
This is the Cartan matrix of the root system $\Phi \subset \mathbb{R}^8$,

$$\Phi = \{\pm \varepsilon_i \pm \varepsilon_j, i \neq j\} \cup \left\{ \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \pm \varepsilon_8) \right\},$$

where, for the roots in the second set, there is always an even number of minus signs. The first set has cardinality $4\binom{8}{2} = 112$, and the second one $\sum_{0 \leq k \leq 4} \binom{8}{2k} = 128$, hence $\#\Phi = 240$. Choosing the linear form $v = N\varepsilon_8^* - \varepsilon_2^* - 2\varepsilon_3^* - 3\varepsilon_4^* - 4\varepsilon_5^* - 5\varepsilon_6^* - 6\varepsilon_7^*$, with N large enough, we get the following sets of positive and simple roots:

$$\begin{aligned}\Phi^+ &= \{\pm\varepsilon_i - \varepsilon_j, i < j < 8\} \cup \{\pm\varepsilon_i + \varepsilon_8, i < 8\} \\ &\quad \cup \left\{ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 + \varepsilon_8) \right\}, \\ \Delta &= \{-\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \varepsilon_6 - \varepsilon_7, \\ &\quad \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\}.\end{aligned}$$

It remains to compute the scalar products of the simple roots to obtain the Dynkin diagram



Taking hyperplane sections one can obtain abstract root systems of type E_7 , and then E_6 . The Dynkin diagram of E_6 has a twofold symmetry which can be used to deduce a root system of type F_4 by *folding*. Similarly, the root system G_2 , which we have already described, can be obtained by folding a root system of type D_4 , the only indecomposable root system with a threefold symmetry.

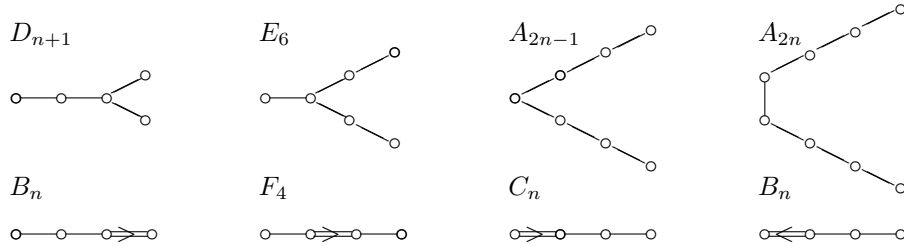
The idea is the following: suppose that the Dynkin diagram of a simple Lie $\mathfrak{g}$ has a symmetry, which means that, once we have fixed a Cartan subalgebra $\mathfrak{h}$ and a set Δ of simple roots, there exists a bijection s of Δ , preserving the Cartan integers. We extend this bijection to an automorphism of $\mathfrak{g}$. To do this, one has first to show that $\mathfrak{g}$ has a Chevalley basis, i.e., a system of vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, and such that the linear map sending X_α to $X_{-\alpha}$, and equal to $-Id$ on $\mathfrak{h}$, is a Lie algebra automorphism of $\mathfrak{g}$. Then we let

$$s(X_\alpha) = X_{s(\alpha)}, \quad s(H_\alpha) = H_{s(\alpha)} \quad \forall \alpha \in \Phi,$$

and check that this defines a Lie algebra automorphism of $\mathfrak{g}$.

Let $\tilde{\mathfrak{g}}$ denote the invariant subalgebra of $\mathfrak{g}$, and $\tilde{\mathfrak{h}}$ denote the invariant subalgebra of $\mathfrak{h}$. By restriction, we have a map $r : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, and we can let $\tilde{\Delta} = r(\Delta)$. Then $\tilde{\Delta} \simeq \Delta / \langle s \rangle$, the automorphism group generated by s . Moreover, one proves that $\tilde{\mathfrak{g}}$ is semi-simple, with Cartan subalgebra $\tilde{\mathfrak{h}}$ and set of simple roots $\tilde{\Delta}$.

One can then deduce the Dynkin diagram of $\tilde{\mathfrak{g}}$ from that of $\mathfrak{g}$. For twofold symmetries, we get the following possibilities for $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$:



The properties of Cartan matrices are fundamental for the classification of root systems: once they have been observed, the classification is just a cumbersome

exercise of Euclidian geometry. There are many variants of solutions, but the main point is that, since the restriction of a scalar product remains a scalar product, any subdiagram of an admissible diagram (i.e., a Dynkin diagram of an irreducible root system) must be admissible. Then one exhibits enough simple nonadmissible diagrams to reduce the possibilities for the admissible diagrams. For example, the two diagrams



are not admissible, and this implies that an admissible diagram has at most triple bonds, each vertex having valency at most three. The upshot is the following theorem due to Killing (circa 1887 – but Killing found two systems of type F_4 , that Cartan recognized to be equivalent).

Theorem 1.1.13. (Classification of root systems) *An indecomposable Dynkin diagram must be of type A , B , C , D , E , F or G .*

This is not the end of the story. Two statements have yet to be proven:

- (1) Every indecomposable abstract root system is the root system of a simple complex Lie algebra.
- (2) A simple complex Lie algebra has a uniquely defined Dynkin diagram and, up to isomorphism, is completely defined by this diagram.

The first point is an *existence theorem*. The case of Dynkin diagrams of type A , B , C , D is easy, since they correspond to the classical Lie algebras. The exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ can be constructed explicitly in many different ways – we will see how later. One can also analyze in more details the structure constants $C_{\alpha\beta}$ and deduce an abstract existence theorem; this was done by Chevalley and Tits. Another method, due to Serre, consists in defining the Lie algebra by generators and relations directly from the Cartan matrix.

The second point is a *unicity theorem*. The construction of the Dynkin diagram of a Lie algebra depends on several choices. First, one has to prove that the Cartan subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are conjugate under automorphisms of $\mathfrak{g}$. Second, one shows that the Weyl group acts transitively on the sets of simple roots of the root system. Conversely, the fact that the structure of $\mathfrak{g}$ is completely determined by the Dynkin diagram, or by the root system, follows from a careful analysis of the structure constants.

Note that there are a few redundancies between the classical Dynkin diagrams. By the unicity theorem we have just mentioned, they detect some isomorphisms between certain small dimensional Lie algebras:

$$\begin{array}{ll}
 B_1 = A_1 & \mathfrak{so}_3 \simeq \mathfrak{sl}_2, \\
 D_2 = A_1 \times A_1 & \mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2, \\
 B_2 = C_2 & \mathfrak{so}_5 \simeq \mathfrak{sp}_4, \\
 D_3 = A_3 & \mathfrak{so}_6 \simeq \mathfrak{sl}_4.
 \end{array}$$

1.2. Representations of semisimple Lie algebras

We have already described the representations of the basic Lie algebra $\mathfrak{sl}_2$. We will see how to describe all the representations of any complex semisimple Lie algebra.

1.2.1. Basic definitions.

DEFINITION 1.2.1. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V , with a morphism of Lie algebras $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$. The vector space V is said to be a $\mathfrak{g}$ -*module*.

Here $\mathfrak{gl}(V)$ stands for the Lie algebra of linear endomorphisms of V , whose Lie bracket is just the usual commutator. We will only consider finite dimensional representation.

DEFINITION 1.2.2. A $\mathfrak{g}$ -module V is *irreducible* if it does not contain any proper submodule. It is *indecomposable* if it cannot be written as the direct sum of two proper submodules.

A basic result to be used later on is:

Proposition 1.2.3. (Schur's lemma) *Let $u : U \rightarrow V$ be a map between irreducible $\mathfrak{g}$ -modules, commuting with the action of $\mathfrak{g}$. Then u is zero, or an isomorphism.*

PROOF. The kernel of u is a $\mathfrak{g}$ -submodule of U , so it must be all of U and $u = 0$, or u is injective. But the image of u is also a $\mathfrak{g}$ -submodule, so it must be all of V and u is an isomorphism. $\square$

It can happen that a $\mathfrak{g}$ -module is indecomposable without being irreducible, but not when $\mathfrak{g}$ is semisimple.

Theorem 1.2.4. *If $\mathfrak{g}$ is a semisimple complex Lie algebra, every finite dimensional $\mathfrak{g}$ -module is completely reducible, i.e., can be decomposed into a direct sum of irreducible submodules.*

This theorem was first proved by H. Weyl, who used the relation to Lie groups: a complex semisimple Lie algebra comes from a complex semisimple Lie group, which can be proved to be the *complexification* of a compact Lie group. This allows one to reduce the problem to representations of compact groups, which behave, as Hurwitz pointed out, very much like representations of finite groups. But then the result was known to follow from averaging methods. This is the famous *unitary trick* of H. Weyl.

Corollary 1.2.5. *The algebra $\text{Der } \mathfrak{g}$ of derivations of a semisimple Lie algebra $\mathfrak{g}$, is isomorphic to $\mathfrak{g}$.*

There is always a map $\mathfrak{g} \rightarrow \text{Der } \mathfrak{g}$, mapping x to $\text{ad}(x)$ – the Jacobi identity precisely means that the adjoint operator $\text{ad}(x)$ is a derivation. Since $\text{Der } \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$, this defines a representation of $\mathfrak{g}$, the *adjoint representation*. Its kernel is the center of $\mathfrak{g}$, which is always an ideal, thus reduces to zero if $\mathfrak{g}$ is semisimple.

1.2.2. The Casimir operator. For any Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$ -module V can be seen as a module over an associative algebra $U(\mathfrak{g})$ called the *universal enveloping algebra*. This is the infinite dimensional algebra defined as the quotient of the tensor algebra $\mathbb{T}(\mathfrak{g}) = \bigoplus_{k \geq 0} \mathfrak{g}^{\otimes k}$, by the relations $X \otimes Y - Y \otimes X = [X, Y]$, for all $X, Y \in \mathfrak{g}$.

Suppose the algebra $\mathfrak{g}$ to be semisimple, and denote by X_i and Y_i two basis of $\mathfrak{g}$ which are dual with respect to the Killing form, and let

$$C = \sum_i X_i \otimes Y_i \in U(\mathfrak{g})$$

denote the *Casimir operator*. It is easy to check that C does not depend on the choice of the basis.

Proposition 1.2.6. *The Casimir operator C commutes with the action of $\mathfrak{g}$ on any module V . In particular, if V is irreducible, C acts on V by some scalar, the Casimir eigenvalue of V .*

PROOF. If $Z \in \mathfrak{g}$, let $[X_i, Z] = \sum_j z_{ij} X_j$. By the invariance of the Killing form, $z_{ij} = K([X_i, Z], Y_j) = K(X_i, [Z, Y_j])$, hence $[Y_j, Z] = -\sum_i z_{ij} X_i$. For any $v \in V$, we have

$$\begin{aligned} C(Zv) &= \sum_i X_i Y_i Zv = \sum_i X_i ([Y_i, Z] + ZY_i)v \\ &= -\sum_{i,j} z_{ji} X_i Y_j v + \sum_i ([X_i, Z] + ZX_i)Y_i v \\ &= -\sum_{i,j} z_{ji} X_i Y_j v + \sum_{i,j} z_{ij} X_j Y_i v + \sum_i ZX_i Y_i v \\ &= Z(Cv). \end{aligned}$$

This proves the first claim. The second one follows from Schur's lemma. $\square$

1.2.3. Weights. From now on we suppose that $\mathfrak{g}$ is simple and we fix a Cartan subalgebra $\mathfrak{t}$. If V is a representation of $\mathfrak{g}$, we can analyse its structure just in the same way that we analyzed the structure of $\mathfrak{g}$: we diagonalize the action of $\mathfrak{t}$ and define the *weight spaces*

$$V_\mu = \{v \in V, \quad h.v = \mu(h)v \quad \forall h \in \mathfrak{t}\}, \quad \mu \in \mathfrak{t}^*.$$

We have that $V = \bigoplus_{\mu \in \mathfrak{t}^*} V_\mu$, and if $V_\mu \neq 0$, μ is a *weight* of the $\mathfrak{g}$ -module V , with *multiplicity* m_μ equal to the dimension of V_μ .

Recall that for each root $\alpha \in \Phi$, we defined in $\mathfrak{g}$ a subalgebra $\mathfrak{g}^{(\alpha)} \simeq \mathfrak{sl}_2$, generated by a triple $(H_\alpha, X_\alpha, X_{-\alpha})$. Since $X_\alpha \in \mathfrak{g}_\alpha$, the Jacobi identity implies that X_α maps V_μ to $V_{\mu+\alpha}$. We can consider V as a $\mathfrak{g}^{(\alpha)}$ -module and apply what we know about $\mathfrak{sl}_2$ -modules. First, H_α acts on V_μ by multiplication by $\mu(H_\alpha)$, which must therefore be an integer, say k . Second, if $v \in V_\mu$ is nonzero, the vectors $v, X_{-\alpha}v, \dots, X_{-\alpha}^k v$ are independent, in particular nonzero. Thus $\mu - i\alpha$ is also a weight of V for $0 \leq i \leq k$, in particular $\mu - \mu(H_\alpha)\alpha = s_\alpha(\mu)$ is a weight of V . Since the operator $X_{-\alpha}^k$ is an isomorphism, as we can easily check on the irreducible $\mathfrak{sl}_2$ -modules, we conclude:

Proposition 1.2.7. *The weights of a $\mathfrak{g}$ -module, and their multiplicities, are invariant under the Weyl group action.*

DEFINITION 1.2.8. Let $\mathcal{P} \subset \mathfrak{t}^*$ be the set of all $\mu \in \mathfrak{t}^*$ such that $\mu(H_\alpha) \in \mathbb{Z} \quad \forall \alpha \in \Phi$. This is a free $\mathbb{Z}$ -module, a *lattice*, of rank the dimension of $\mathfrak{t}$, i.e., the rank of $\mathfrak{g}$; it is called the *weight lattice*.

If a set Δ of simple roots is fixed, a basis of $\mathcal{P}$ is given by the *fundamental weights* $\omega_\alpha, \alpha \in \Delta$, defined by the condition that $\omega_\alpha(H_\beta) = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. The nonnegative integer linear combinations of the fundamental weights are the *dominant weights*, whose set we denote by $\mathcal{P}^+ \subset \mathcal{P}$.

REMARK 1.2.9. The Cartan matrix gives the simple roots in terms of the fundamental weights, $\alpha = \sum_{\beta \in \Delta} c_{\alpha\beta} \omega_{\beta}$.

Note that the weights of the adjoint representation are 0 (with multiplicity equal to the rank of $\mathfrak{g}$), and the roots (with multiplicity one). In particular $\mathcal{P}$ contains the *root lattice* $\mathcal{Q}$ generated by the roots.

Let us come back to the $\mathfrak{g}$ -module V , and let us say that a weight λ is a *highest weight* of V if no weight of V has the form $\lambda + \alpha$ for a positive root $\alpha \in \Phi^+$. If $v \in V_{\lambda}$ is nonzero, we claim that the subspace of V generated by the vectors of the form $X_{-\alpha_{i_p}} \cdots X_{-\alpha_{i_1}} v$, for any choice of positive roots $\alpha_1, \dots, \alpha_p \in \Phi^+$, is a $\mathfrak{g}$ -submodule of V (check this!). If V is irreducible, this implies that (1) λ has multiplicity one, (2) every weight of V is of the form $\lambda - \sum_{\alpha \in \Delta} n_{\alpha} \alpha$ for some coefficients $n_{\alpha} \in \mathbb{Z}_+$. Since the set of weights is W -invariant, this implies that λ must be dominant.

Theorem 1.2.10. *An irreducible $\mathfrak{g}$ -module is, up to isomorphism, uniquely determined by its highest weight $\lambda \in \mathcal{P}^+$; we denote it by V_{λ} .*

Conversely, for each dominant weight $\lambda \in \mathcal{P}^+$, there exists an irreducible $\mathfrak{g}$ -module V_{λ} with highest weight λ .

We have precise information on the weights of V_{λ} .

Proposition 1.2.11. *The set of weights of V_{λ} is equal to the intersection of the convex hull of $W\lambda$, the set of images of $\lambda \in \mathcal{P}^+$ by the action of the Weyl group, with the translate $\lambda + \mathcal{Q}$ of the root lattice.*

EXAMPLE 1.2.12. The adjoint representation of a simple complex Lie algebra $\mathfrak{g}$ is irreducible. This implies that among the positive roots, there is one root ψ which is the highest weight, hence such that for any positive root β , the sum $\psi + \beta$ is not a root. It turns out that for all simple Lie algebras, except $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the highest root is a fundamental weight. Said otherwise, the adjoint representation is fundamental.

The unicity assertion of Theorem 1.2.10 is easy to prove: if we have two irreducible modules V and V' with the same highest weight, we choose nonzero vectors v and v' in the corresponding weight spaces. Denote by $V'' \subset V \oplus V'$ the irreducible $\mathfrak{g}$ -module generated by (v, v') . The restrictions to V'' of the projections to V and V' are certainly nonzero, so by Schur's lemma they must be isomorphisms. Thus $V \simeq V'' \simeq V'$.

The existence assertion is much more delicate, and infinite dimensional modules are in general used to prove it. One way to avoid this (that was E. Cartan initial line of ideas), is to construct first the *fundamental representations*, i.e., the irreducible $\mathfrak{g}$ -modules whose highest weights are the fundamental ones. All other representations can be deduced from these if we observe that in the tensor product $V_{\lambda} \otimes V_{\mu}$ of two irreducible $\mathfrak{g}$ -modules, the weight $\lambda + \mu$ is a highest weight. Therefore, the $\mathfrak{g}$ -module generated by the tensor product $v_{\lambda} \otimes v_{\mu}$ of two highest weight vectors is an irreducible module $V_{\lambda+\mu}$ of highest weight $\lambda + \mu$. (This is called the *Cartan product*). Since any dominant weight is a sum of fundamental ones, this allows one to construct any irreducible $\mathfrak{g}$ -module inside some tensor product of fundamental representations.

1.2.4. Representations of $\mathfrak{sl}_n$. Let us complete this program in the case of $\mathfrak{sl}_n$. An obvious representation is given by the natural action on $\mathbb{C}^n$, which is obviously irreducible. The weights are the ε_i , $1 \leq i \leq n$, and with the choice of positive roots we made in the first lecture, the highest weight is $\omega_1 = \varepsilon_1$, the first fundamental weight (check this!).

More generally, we can look at the induced action of $\mathfrak{sl}_n$ on the wedge powers $\wedge^k \mathbb{C}^n$, defined for $X \in \mathfrak{sl}_n$ by

$$X(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^k v_1 \wedge \cdots \wedge Xv_i \wedge \cdots \wedge v_k.$$

Here the weights are the sums $\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$, and the highest one is $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, which is again a fundamental weight. We have thus identified the fundamental representations to be the wedge powers $\wedge^k \mathbb{C}^n$, with $1 \leq k \leq n-1$, and we conclude that any irreducible $\mathfrak{sl}_n$ -module can be obtained as a submodule of some tensor product $(\mathbb{C}^n)^{\otimes a_1} \otimes (\wedge^2 \mathbb{C}^n)^{\otimes a_2} \otimes \cdots \otimes (\wedge^{n-1} \mathbb{C}^n)^{\otimes a_{n-1}}$.

The representations of $\mathfrak{sl}_n$ were actually first constructed by I. Schur in a different way. The idea was to use the symmetric groups $\mathfrak{S}_k$. If $V \simeq \mathbb{C}^n$, $\mathfrak{S}_k$ acts on $V^{\otimes k}$ by permuting the factors: $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. If A is any finite dimensional $\mathfrak{S}_k$ -module, we can define

$$\text{Hom}_{\mathfrak{S}_k}(A, V^{\otimes k}) := \{u \in \text{Hom}(A, V^{\otimes k}), \quad u(\sigma a) = \sigma u(a) \quad \forall \sigma \in \mathfrak{S}_k\}.$$

The representations of $\mathfrak{S}_k$ are well-understood: they are completely reducible, and the irreducible representations are classified by *partitions* of k . Indeed, choose such a partition π , a nonincreasing sequence of positive integers $\pi_1 \geq \cdots \geq \pi_l$, with $\pi_1 + \cdots + \pi_l = k$. Such a partition is often represented by a *Young diagram* D_π as follows: the lengths of the lines are the π_i 's, so in the example below $\pi = (4, 3, 1)$ is a partition of $k = 8$:

1	4	6	8
2	5	7	
3			

Number the boxes of the Young diagram D_π from 1 to k , in an arbitrary way, to obtain a *tableau* T of shape π . If $x_1, \dots, x_k$ are indeterminates, we associate to the tableau T the polynomial

$$P_T = \prod_{i <_T j} (x_i - x_j),$$

where the relation $i <_T j$ means that i and j appear in the same column of D_π , with i above j . Of course, the group $\mathfrak{S}_k$ acts on the set of tableaux of shape π , as well as on the indeterminates $x_1, \dots, x_k$, and $P_{\sigma T} = \sigma P_T$.

Theorem 1.2.13. *The space of polynomials generated by the P_T 's, where T describes the set of tableaux of shape π , is an irreducible $\mathfrak{S}_k$ -module, denoted by $[\pi]$ and called a *Specht module*.*

A basis of $[\pi]$ is given by the polynomials P_T , where T describes the set of standard tableaux of shape π .

A tableau is *standard* when it is numbered by consecutive integers, starting from 1, in such a way that its labels increase from left to right on each line, and from top to bottom on each column.

Let us come back to our vector space V , and let $S_\pi V := \text{Hom}_{\mathfrak{S}_k}([\pi], V^{\otimes k})$, the π -th Schur power of V . Note that $GL(V)$ acts on $V^{\otimes k}$ by $g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$. Since this action commutes with the action of the symmetric group $\mathfrak{S}_k$, there is an induced action of $GL(V)$ on $S_\pi V$. This is also true for the Lie algebra $\mathfrak{gl}(V) \supset \mathfrak{sl}(V) \simeq \mathfrak{sl}_n$.

Theorem 1.2.14. *The Schur powers $S_\pi V$ are irreducible $\mathfrak{sl}(V)$ -modules (or zero), and every irreducible $\mathfrak{sl}(V)$ -module can be obtained that way.*

Theorem 1.2.15. (Schur duality) *For each $k > 0$, the canonical map*

$$\bigoplus_{\pi \text{ partition of } k} [\pi] \otimes S_\pi V \longrightarrow V^{\otimes k}$$

is an isomorphism of $\mathfrak{S}_k \times GL(V)$ -modules.

EXAMPLE 1.2.16. If $\pi = (k)$, D_π has only one line, and we have $P_T = 1$ for every tableau T of shape π . Thus $[k]$ is just the trivial representation, and $S_\pi V$ is the space of symmetric tensors in $V^{\otimes k}$, the k -th symmetric power $S^k V$.

EXAMPLE 1.2.17. If $\pi = (1^k)$, D_π has only one column, and we have

$$P_T = \pm \prod_{1 \leq i < j \leq k} (x_i - x_j)$$

for every tableau T of shape π . Hence $[1^k]$ is again one-dimensional but is not trivial: it is the *sign representation* of $\mathfrak{S}_k$. Thus $S_\pi V$ is the space of skew-symmetric tensors in $V^{\otimes k}$, the k -th wedge power $\wedge^k V$.

A vector in $S_\pi V = \text{Hom}_{\mathfrak{S}_k}([\pi], V^{\otimes k})$ is defined by the image of a single generator P_T of $[\pi]$, since the images of the other ones are then deduced from the $\mathfrak{S}_k$ -action. For example, choose for T the tableau numbered column after column, from top to bottom. If σ is a transposition of two labels appearing in the same column of T , we have $\sigma P_T = -P_T$. Thus, the image of P_T must be skew-symmetric with respect to these two labels, and we recover the fact that $S_\pi V$ is contained in a product of wedge powers, namely $\wedge^{l_1} V \otimes \cdots \otimes \wedge^{l_s} V$, where the l_i denote the lengths of the columns of D_π .

Let us now choose a basis $v_1, \dots, v_n$ of V . Say that a tableau S numbered by integers between 1 and n is *semistandard* if its labels never decrease from left to right on each line, and increase from top to bottom on each column. One can prove that there exists a map $\phi_S \in S_\pi V$ mapping P_T to the vector v_S defined as the tensor product of the $v_{j_1} \wedge \cdots \wedge v_{j_{l_i}} \in \wedge^{l_i} V \subset V^{\otimes l_i}$.

Proposition 1.2.18. *The maps ϕ_S give a basis of $S_\pi V$, whose dimension is therefore equal to the number of semistandard tableaux of shape π , labeled by integers not greater than the dimension of V .*

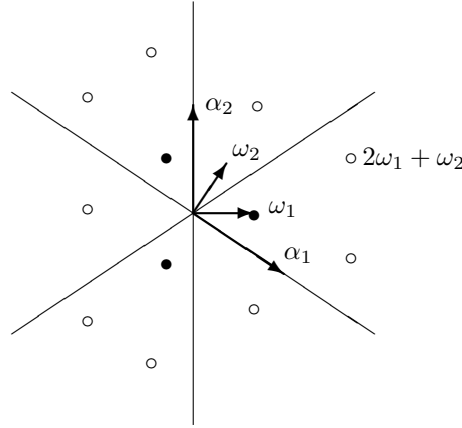
A consequence is that $S_\pi V \neq 0$ if and only if the number l of parts of π does not exceed the dimension of V .

We can also deduce the weights of $S_\pi V \neq 0$ as an $\mathfrak{sl}(V)$ -module, since the Cartan subalgebra of matrices that are diagonal in the basis of V that we have chosen, has a diagonal action in the basis of $S_\pi V$ provided by the ϕ_S . And the weight of ϕ_S is obviously $\omega_S = \sum_i S_i \varepsilon_i$, if we denote by S_i the number of labels equal to i in S (recall that the trace condition implies that $\varepsilon_1 + \cdots + \varepsilon_n = 0$). The

highest of these weights is obtained when S is the tableau numbered by i 's on the whole i -th line: we get

$$\omega_S = \sum_i \pi_i \varepsilon_i = \sum_j (\pi_j - \pi_{j+1}) \omega_j.$$

EXAMPLE 1.2.19. Let $n = 3$ and $\pi = (3, 1)$, so that $S_\pi \mathbb{C}^3$ is the irreducible $\mathfrak{sl}_3$ -module with highest weight $2\omega_1 + \omega_2$. It is straightforward to list the semistandard tableaux of shape π and to check that $S_\pi \mathbb{C}^3$ has dimension 15. Its weights are as follows, with a $\circ$ for multiplicity one and a $\bullet$ for multiplicity two:



The highest weight has 6 different images under the Weyl group action. On the boundary of their convex hulls, there are three other weights with multiplicity one. And there remain three other weights in the interior of the convex hull, with multiplicity two.

1.2.5. Representations of other classical Lie algebras. For the other classical Lie algebras we can identify the fundamental representations, which in principle allows the construction of all irreducible representations.

We begin with the symplectic algebra $\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$. The natural representation on $\mathbb{C}^{2n}$ is of course irreducible, and fundamental. But the induced action on its wedge powers is not irreducible. Recall that $\mathfrak{sp}_{2n}$ is defined by the condition that ${}^t XJ + JX = 0$, where J denotes an invertible skew-symmetric matrix. Such a matrix can be interpreted as a nondegenerate skew-symmetric bilinear form, i.e., a *symplectic form* θ on $\mathbb{C}^{2n}$, and $X \in \mathfrak{sl}_{2n}$ belongs to $\mathfrak{sp}_{2n}$ if and only if

$$\theta(Xx, y) + \theta(x, Xy) = 0 \quad \forall x, y \in \mathbb{C}^{2n}.$$

Being non-degenerate, the symplectic form $\theta \in \wedge^2(\mathbb{C}^{2n})^*$ identifies $\mathbb{C}^{2n}$ with its dual. In particular, θ defines a dual tensor $\theta^* \in \wedge^2 \mathbb{C}^{2n}$, and the line generated by θ^* is a submodule of $\wedge^2 \mathbb{C}^{2n}$, which is therefore reducible. Nevertheless, define

$$\wedge^{(k)} \mathbb{C}^{2n} = \{\psi \in \wedge^k \mathbb{C}^{2n}, \quad \psi \wedge (\theta^*)^{n-k+1} = 0\}.$$

Proposition 1.2.20. *For $1 \leq k \leq n$, $\wedge^{(k)} \mathbb{C}^{2n}$ is an irreducible $\mathfrak{sp}_{2n}$ -module, and this is the complete list of fundamental representations of $\mathfrak{sp}_{2n}$.*

As $\mathfrak{sp}_{2n}$ -modules, the ordinary wedge powers $\wedge^k \mathbb{C}^{2n}$ decompose as

$$\wedge^k \mathbb{C}^{2n} = \bigoplus_{l \geq 0} \wedge^{k-2l} \mathbb{C}^{2n}.$$

The Dynkin diagram gives the following picture:

$$\begin{array}{ccccccc} \mathbb{C}^{2n} & & & & & & \wedge^{(n-1)} \mathbb{C}^{2n} \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & \wedge^{(2)} \mathbb{C}^{2n} & & & & & \wedge^{(n)} \mathbb{C}^{2n} \end{array}$$

The case of the orthogonal Lie algebras $\mathfrak{so}_n$ has some peculiarities. Again we have the natural representation $\mathbb{C}^n$, and its wedge powers $\wedge^k \mathbb{C}^n$ can be proved to be irreducible, and fundamental, for $1 \leq k \leq \frac{n}{2} - 2$. Then something unexpected happens: $\wedge^{m-1} \mathbb{C}^{2m}$ and $\wedge^m \mathbb{C}^{2m+1}$ are irreducible but not fundamental, and $\wedge^m \mathbb{C}^{2m}$ is not even irreducible! This phenomenon is due to the existence of the *spinor representations*.

REMARK 1.2.21. Note the general fact that if we know the fundamental representation V attached to a vertex which is at an end of the Dynkin diagram, we can construct other fundamental representations by taking wedge powers. If our vertex is the extremity of a chain of length k in the Dynkin diagram, with only simple bonds, then one can show that the fundamental representations attached to the vertices of that chain, can be obtained as direct summands V_l of the wedge powers $\wedge^l V$, $1 \leq l \leq k+1$. This was observed by Cartan who used it to give the first construction of all fundamental representations of all complex simple Lie algebras.

1.2.6. Weyl's dimension formula. Hermann Weyl was the first to obtain general formulas for the dimensions and multiplicities of an irreducible representation of a simple complex Lie algebra. To state his results, let us define the *character* of a $\mathfrak{g}$ -module V to be the formal sum

$$\text{ch}(V) = \sum_{\mu \in \mathcal{P}} (\dim V_\mu) e^\mu \in \mathbb{Z}[\mathcal{P}].$$

The e^μ are just formal symbols, but the exponential notation indicates that we can multiply them according to the rule $e^\mu e^\nu = e^{\mu+\nu}$. The character is then well-behaved under the natural operations on $\mathfrak{g}$ -modules:

$$\text{ch}(V \oplus V') = \text{ch}(V) + \text{ch}(V') \quad \text{and} \quad \text{ch}(V \otimes V') = \text{ch}(V) \text{ch}(V').$$

If $w \in W$ is an element of the Weyl group, w will permute the roots of $\mathfrak{g}$ and we can define its *length* to be

$$l(w) = \#\{\alpha \in \Phi^+, w(\alpha) \in \Phi^-\}.$$

Let ρ denote the sum of the fundamental weights, the smallest strictly dominant weight. One can show that $2\rho = \sum_{\alpha \in \Phi^+} \alpha$.

Theorem 1.2.22. (Weyl's character formula). *The character of an irreducible $\mathfrak{g}$ -module V_λ is given by the formula*

$$\text{ch}(V_\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}.$$

Corollary 1.2.23. (Weyl's dimension formula). *The dimension of an irreducible $\mathfrak{g}$ -module V_λ is given by the formula*

$$\dim V_\lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

EXAMPLE 1.2.24. We come back to the representations $S_\pi \mathbb{C}^n$ of $\mathfrak{sl}_n$. The Weyl group here is the symmetric group $\mathfrak{S}_n$, and the length that we have defined is the usual length of a permutation $\sigma \in \mathfrak{S}_n$:

$$l(\sigma) = \#\{i < j, \sigma(i) > \sigma(j)\}.$$

The sums over the Weyl group that appear in Weyl's character formula can be interpreted as *determinants*, and the Weyl character formula can be rewritten as follows. Define the *Schur function*

$$s_\pi(x_1, \dots, x_n) = \frac{\det(x_i^{\pi_j + n - j})_{1 \leq i, j \leq n}}{\det(x_i^{n - j})_{1 \leq i, j \leq n}}.$$

This is in fact a polynomial, since the denominator of the formula is the usual Vandermonde determinant $\det(x_i^{n - j})_{1 \leq i, j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Obviously this product divides any skew-symmetric polynomial, and in particular the numerator of the formula.

Weyl's character and dimension formulas can now be written as

$$\begin{aligned} \text{ch}(S_\pi \mathbb{C}^n) &= s_\pi(e^{\varepsilon_1}, \dots, e^{\varepsilon_n}), \\ \dim S_\pi \mathbb{C}^n &= \prod_{1 \leq i < j \leq n} \frac{\pi_i - \pi_j + j - i}{j - i}. \end{aligned}$$

1.3. Linear algebraic groups

DEFINITION 1.3.1. An *affine algebraic group* is a group G , endowed with a structure of affine variety over some field, such that the group law $G \times G \rightarrow G$, mapping (x, y) to xy^{-1} , is algebraic.

An affine variety, say over the complex numbers, is fully determined by the algebra $\mathbb{C}[G]$ of regular functions. The group multiplication induces a morphism

$$\mathbb{C}[G] \longrightarrow \mathbb{C}[G \times G] = \mathbb{C}[G] \otimes \mathbb{C}[G].$$

Group operations are well-behaved algebraically: if ϕ is any morphism of affine algebraic groups, not only its kernel, but also its image are affine algebraic groups. Another important property is that an affine algebraic group G has a unique irreducible component containing the identity element. This component G° is a closed normal subgroup of finite index, and coincides with the connected component of the identity.

Theorem 1.3.2. Any affine algebraic group is isomorphic to a closed subgroup of $GL_n(\mathbb{C})$ for some integer n .

PROOF. The algebra $\mathbb{C}[G]$ is generated by some finite dimensional subspace V , which can be chosen to be invariant by left translations (prove this !). Then we get a map λ from G to $GL(V)$ by letting

$$(\lambda(g)f)(x) = f(gx).$$

This map is algebraic, a group morphism, and a closed embedding. $\square$

We could therefore have considered directly *linear algebraic groups*, which are defined as the closed subgroups of the $GL(V)$.

1.3.1. Algebraic groups and Lie algebras. One of the possible algebraic definitions of the Lie algebra of an affine algebraic group G makes use of the algebra $D(G)$ of *derivations* of $\mathbb{C}[G]$, i.e., maps

$$d : \mathbb{C}[G] \rightarrow \mathbb{C}[G], \quad d(fg) = fd(g) + gd(f).$$

Let $\lambda(g)$ denote the left translation by g on $\mathbb{C}[G]$, defined as in the proof of the previous Theorem. There is an induced action on $D(G)$ by $\lambda(g)(d) = \lambda(g) \circ d \circ \lambda(g^{-1})$, and we denote by $L(G)$ the space of *left-invariant derivations*, defined by the condition that $\lambda(g)(d) = d$ for all $g \in G$.

The commutator $[d, d'] = dd' - d'd$ endows $D(G)$ with a Lie algebra structure, and $L(G)$ is obviously a subalgebra. This structure is transferred to the tangent space of G at the identity element e , by the map

$$\begin{aligned} \theta : L(G) &\longrightarrow T_e G \simeq \text{Der}(\mathcal{O}_{G,e}, \mathbb{C}) \\ d &\longmapsto \theta(d)(f) = (df)(e). \end{aligned}$$

Here $\mathcal{O}_{G,e}$ denotes the algebra of rational functions on G that are defined at e . Locally around e such a rational function can be written as a quotient of two regular functions, to which we can apply a derivation d by the usual formula for the derivative of a quotient. Then we evaluate the resulting function at e .

Proposition 1.3.3. The map θ is a vector space isomorphism.

PROOF. If $\delta \in T_e G$ and $f \in \mathbb{C}[G]$, let $\delta^*(f)(x) = \delta(\lambda(x^{-1})f)$. Prove that this defines an inverse for θ . $\square$

We denote by $\mathfrak{g}$ the vector space $T_e G$ endowed with the resulting Lie algebra structure. If ϕ is any algebraic morphism between affine algebraic groups, it is a formal verification that the tangent map at the identity is a Lie algebra morphism.

EXAMPLE 1.3.4. If $g \in G$, define the map $\text{Int}(g) : G \rightarrow G$ by $\text{Int}(g)(x) = gxg^{-1}$. The tangent map is denoted $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$, and since $\text{Ad}(g) \circ \text{Ad}(h) = \text{Ad}(gh)$, we get a morphism

$$\text{Ad} : G \longrightarrow \text{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$$

called the *adjoint representation*. One can check that the tangent map to Ad is the map $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ defined by $\text{ad}(x)(y) = [x, y]$. Indeed, the Jacobi identity on $\mathfrak{g}$ is equivalent to the fact that the operators $\text{ad}(x)$ are derivations.

1.3.2. Homogeneous spaces. Let H be some closed subgroup of an affine algebraic group G . We would like to endow the set G/H of right H -cosets in G , with an algebraic structure. The resulting algebraic variety will be a *homogeneous space*, i.e., an algebraic variety with a transitive group action.

Theorem 1.3.5. (Chevalley) *There exists an embedding $G \subset GL(V)$, and a line L in the finite dimensional vector space V , such that*

$$H = \{g \in G, \quad gL = L\}.$$

PROOF. Let $I \subset \mathbb{C}[G]$ denote the ideal of the closed subgroup H . Include a finite set of generators of I into some finite dimensional subspace B of $\mathbb{C}[G]$, stable under left-translations, and let $A = B \cap I$. Then G acts on B – check that the stabilizer of A is equal to H . To prove the claim, there just remains to replace A by the line $L = \wedge^a A$ inside $V = \wedge^a B$, where $a = \dim A$. $\square$

The G -orbit of L , i.e., the set $X = G.L = \{gL, g \in G\} \subset \mathbb{P}V$, is then a *quasi-projective variety* with a G -action, with a base point $x = [L]$ whose stabilizer is equal to H , and such that the fibers of the map $G \rightarrow X, g \mapsto gx$, are axactly the H -cosets.

To call this space the *quotient* of G by H , we need to prove that it is canonical in some sense. One can show that it has the following characteristic property: for each G -homogeneous space Y , with a base point y whose stabilizer contains H , there exists a G -equivariant algebraic morphism $(X, x) \rightarrow (Y, y)$. A homogeneous space, like X , with this property is unique up to isomorphism (in characteristic zero), and this is what we call a quotient of G by H .

The existence of quotients has a useful consequence: if G is a connected affine algebraic group, there is a bijective correspondence between the closed connected subgroups of G and their Lie algebras, which are subalgebras of $\mathfrak{g} = \text{Lie}(G)$. But beware that a Lie subalgebra of $\mathfrak{g}$ is not always the Lie algebra of a closed subgroup of G !

1.3.3. Solvable groups. For any group G , one can define the *derived subgroup* $D^1(G) = [G, G]$ to be the normal subgroup generated by the commutators $[g, h] = g^{-1}h^{-1}gh$. If G is an affine algebraic group, then $D(G)$ is a closed subgroup. More generally, we define the derived series inductively, by

$$D^{n+1}(G) = [D^n(G), D^n(G)].$$

DEFINITION 1.3.6. The group G is *solvable* if $D^n(G) = 1$ for some $n \geq 0$.

Solvable groups have a very nice geometric property:

Theorem 1.3.7. (Borel's fixed point theorem) *Let G be a connected solvable affine algebraic group, and let X be a complete G -variety. Then X has a fixed point.*

This has the following classical consequence. Suppose G is a solvable closed connected subgroup of $GL(V)$. There is an induced action of G on the set of complete flags of subspaces of V . Since this *complete flag variety* is projective, G has a fixed point. Consequence:

Corollary 1.3.8. (Lie-Koelchin's theorem) *Let $G \subset GL(V)$ be a solvable closed connected subgroup. Then one can find a basis of V in which G consists of triangular matrices.*

Proof of Borel's fixed point theorem. Let $X(G)$ denote the set of fixed points of G in X . We prove that $X(G) \neq \emptyset$ by induction on the dimension of G .

The derived group $[G, G]$ is again a connected solvable affine algebraic group, of smaller dimension than G . Therefore the fixed point set $X([G, G])$ is a non empty closed subset of X . This is a complete variety, on which G acts in such a way that the stabilizer G_x of a point x always contains $[G, G]$. In particular, G_x is a closed normal subgroup, and this implies that the quotient space G/G_x is an affine algebraic group (this is the algebraic version of the classical fact that the quotient of any abstract group by a normal subgroup has an abstract group structure). In characteristic zero, G/G_x can be identified with the orbit Gx of x , which is therefore affine.

But an orbit $\mathcal{O}$ is always *constructible*, so that its boundary $\partial\mathcal{O} = \overline{\mathcal{O}} - \mathcal{O}$ has strictly smaller dimension, and is of course G -stable. This implies in particular that an orbit of minimal dimension must be closed. In our particular situation, we can therefore choose x such that Gx is closed, hence complete. Being complete, affine, and connected since G is connected, it must be a point. Thus $x \in X(G) \neq \emptyset$. $\square$

DEFINITION 1.3.9. Let G be an affine algebraic group. A closed subgroup B is a *Borel subgroup* if it is a maximal connected solvable subgroup. A closed subgroup P is a *parabolic subgroup* if the quotient G/P is a projective variety.

Theorem 1.3.10. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup. Moreover, Borel subgroups of G are all conjugate.*

PROOF. If P is parabolic in G , a Borel subgroup B acts by left translations on the projective variety G/P , and has a fixed point xP by Borel's fixed point theorem. Then $BxP \subset xP$, which implies that P contains the Borel subgroup $x^{-1}Bx$ of G .

Conversely, if a closed subgroup P contains a Borel subgroup B , we have a surjective map $G/B \rightarrow G/P$. If we can prove that G/B is projective, this will imply that G/P is complete, hence projective since we know by Chevalley's theorem that it is quasi-projective. Embed G in some $GL(V)$. Then G acts on the variety of complete flags of V , and some orbit of this action, say G/H , must be closed, hence projective. This means that H is parabolic, hence contains some conjugate $x^{-1}Bx$ of B . But by definition H is contained in the stabilizer of some complete flag of V ; this stabilizer is a group of triangular matrices, hence is solvable, so H itself is solvable. By the definition of Borel subgroups, we must have $H^\circ = x^{-1}Bx$. Then

the quotient map $G/B \rightarrow G/xHx^{-1}$ is finite, and since G/xHx^{-1} is projective, this implies that G/B is projective.

So, Borel subgroups are parabolic. Applying the first part of the proof, we conclude that they are all conjugate. $\square$

1.3.4. Semi-simple algebraic groups.

DEFINITION 1.3.11. If G is an affine algebraic group, define its *radical* $R(G)$ to be the maximal connected solvable normal subgroup. G is *semi-simple* if $R(G) = 1$.

Note that the center $Z(G)$ is a solvable normal subgroup, so that $Z(G)^\circ \subset R(G)$. In particular, a semi-simple group has finite center.

Proposition 1.3.12. *A connected affine algebraic group G is semi-simple if and only if its Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is semi-simple.*

PROOF. If G is not semi-simple, the last non zero term N of the derived series $D^n(R(G))$ is a normal connected abelian subgroup. Its Lie algebra is an abelian ideal $\mathfrak{n}$ of $\mathfrak{g}$, which cannot be semi-simple (apply Cartan's criterion).

Conversely, suppose that $\mathfrak{g}$ is not semi-simple, hence contains an abelian ideal $\mathfrak{n}$. The group G acts on $\mathfrak{g}$ through the adjoint representation and we can let $H = C_G(\mathfrak{n})^\circ$, the connected component of the stabilizer of $\mathfrak{n}$. Then $\mathfrak{t} = \text{Lie}(H)$ is an ideal (by Jacobi).

Let $M = \{g \in G, \text{Ad}(g)\mathfrak{t} = \mathfrak{t}\}$. Its Lie algebra is

$$\mathfrak{m} = \{x \in \mathfrak{g}, \text{ad}(x)(\mathfrak{t}) \subset \mathfrak{t}\} = \mathfrak{g},$$

since $\mathfrak{t}$ is an ideal. Since G is connected, this implies that $M = G$. Thus H and $g^{-1}Hg$ have the same Lie algebra for all $g \in G$, hence they are equal, which means that H is normal. Then its center $Z(H)$ is also normal, and its Lie algebra is the center of $\mathfrak{t}$, which contains $\mathfrak{n}$. This implies that $Z(H)^\circ$ is a non trivial connected abelian normal subgroup of G , so that G is not semi-simple. $\square$

Another important property is that any semi-simple Lie algebra is the Lie algebra of a semi-simple affine algebraic group.

Proposition 1.3.13. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then $G = \text{Aut}(\mathfrak{g})^\circ$ is a connected semi-simple affine algebraic group such that $\text{Lie}(G) = \mathfrak{g}$.*

PROOF. The group $G = \text{Aut}(\mathfrak{g})^\circ$ is clearly a closed subgroup of $GL(\mathfrak{g})$, and its Lie algebra is easily seen to coincide with the derivation algebra $\text{Der}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. So we just need to prove the following property:

Lemma 1.3.14. *If $\mathfrak{g}$ is a semi-simple Lie algebra, then every derivation is inner, i.e., $\text{Der}(\mathfrak{g}) = \mathfrak{g}$.*

Proof of the Lemma. Recall that we have the adjoint map $\mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$, which is injective since a semi-simple Lie algebra has trivial center. Now let d be any non zero derivation. Letting $[x, d] = -d(x)$, we get a natural $\mathfrak{g}$ -module structure on $\mathfrak{g} \oplus \mathbb{C}d$. Since $\mathfrak{g}$ is semi-simple, this $\mathfrak{g}$ -module is completely reducible, thus can be decomposed into the direct sum of $\mathfrak{g}$ with a line generated by some vector $d + x$, $x \in \mathfrak{g}$. But $\mathfrak{g}$ being semi-simple, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and any one-dimensional representation is trivial. This means that for all $y \in \mathfrak{g}$, $[y, d] + [y, x] = 0$, i.e., $d(y) = [x, y]$ and $d = \text{ad}(x)$ is inner. $\square$

This very good correspondence between semi-simple groups and algebras being established, we would like to use what we learned about semi-simple Lie algebras to understand the structure of semi-simple groups.

1.3.5. Maximal tori. First we look for the counterparts of Cartan subalgebras, which will be *maximal tori*. A *torus* is a copy of $(\mathbb{G}_m)^k$, where $\mathbb{G}_m \simeq \mathbb{C}^*$ denotes the multiplicative group. It has the property of being *diagonalizable*, which means that every (finite dimensional) representation can be diagonalized. Conversely, every connected diagonalizable group is a torus.

A difficult part of the theory is to prove that in any affine algebraic group, *all maximal tori are conjugate*. The fact that Borel subgroups are conjugate reduces to the case of solvable groups, which requires a careful analysis.

Once we have proved that, the following definition makes sense.

DEFINITION 1.3.15. The *rank* of an affine algebraic group is the dimension of its maximal tori.

The Lie algebra of any maximal torus T is a Cartan subalgebra $\mathfrak{t}$: indeed, it is an abelian subalgebra, all of whose elements are semi-simple, since a torus is diagonalizable, and the maximality is inherited from that of the torus.

Then we decompose the adjoint action of T on $\mathfrak{g}$. If G is semi-simple, this action can be diagonalized as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha.$$

The slight difference with the decomposition of section 1.1.2 is that the roots α are not elements of $\mathfrak{t}^*$, but of the *character group* $X^*(T) = \text{Hom}(T, \mathbb{G}_m)$ of the maximal torus. Passing to the tangent map we recover the decomposition of the adjoint action of $\mathfrak{t}$, the root space decomposition.

1.3.6. Root subgroups. The next step is to interpret the roots inside the group. At the Lie algebra level, to a root α we can associate $\mathfrak{t}_\alpha = \text{Ker}(\alpha) \subset \mathfrak{t}$, whose centralizer is $\mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{t}_\alpha \oplus \mathfrak{g}^{(\alpha)}$. At the group level, beginning with $\alpha \in X^*(T)$ we consider $T_\alpha = (\text{Ker } \alpha)^o$. The Lie algebra of $C_G(T_\alpha)^o/T_\alpha$ is then a copy of $\mathfrak{sl}_2$. A more precise statement is the following:

Proposition 1.3.16. *A semi-simple affine algebraic group of rank one is isomorphic either to $SL_2(\mathbb{C})$ or to $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$.*

An immediate consequence is that the centralizer $C_G(T_\alpha)^o$ has exactly two Borel subgroups B_α and $B_{-\alpha}$ containing T , whose Lie algebras are $\mathfrak{t} \oplus \mathfrak{g}_\alpha$ and $\mathfrak{t} \oplus \mathfrak{g}_{-\alpha}$, respectively. The set U_α of unipotent elements in B_α is then a connected subgroup (use Lie-Koelchin's theorem), and $\text{Lie}(U_\alpha) = \mathfrak{g}_\alpha$.

One can prove that a one-dimensional connected affine algebraic group must be the multiplicative group $\mathbb{G}_m$, or the additive group $\mathbb{G}_a$. Only the latter is unipotent, thus $U_\alpha \simeq \mathbb{G}_a$.

EXAMPLE 1.3.17. Let $G = SL_n(\mathbb{C})$. The subgroup T of diagonal matrices is a maximal torus, and the characters of T involved in the root space decomposition are given by $\alpha(h) = h_i h_j^{-1}$ for some $i \neq j$, where h_i denotes the i -th diagonal coefficient of $h \in T$. The corresponding root subgroup of G is

$$U_\alpha = \{I + tE_{i,j}, \quad t \in \mathbb{C}\} \simeq \mathbb{G}_a,$$

if $E_{i,j}$ denotes the matrix whose only non zero coefficient is a 1, at the intersection of the i -th line and j -th column.

Note that in the previous example,

$$h(I + tE_{i,j})h^{-1} = I + h_i h_j^{-1} t E_{i,j} = I + \alpha(h) t E_{i,j}.$$

This is a general fact that the group isomorphisms $x_\alpha : \mathbb{G}_a \rightarrow U_\alpha$ are such that $h x_\alpha(t) h^{-1} = x_\alpha(\alpha(h)t)$.

If G is connected, it is generated by the root groups. We can also construct Borel subgroups from the root groups. For example, the group of triangular matrices is a Borel subgroup of $SL_n(\mathbb{C})$, and its Lie algebra is generated by the Lie algebra of the diagonal torus, and the positive root spaces. This suggests to recover a Borel subgroup B , containing a maximal torus T , from the Lie algebra

$$\mathfrak{b} = \text{Lie}(B) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(B)} \mathfrak{g}_\alpha.$$

One can prove that $\Phi(B)$ is a set of positive roots in the root system Φ of $\mathfrak{g}$.

Proposition 1.3.18. *Let $\Phi(B) = \{\alpha_1, \dots, \alpha_N\}$.*

The map $T \times U_{\alpha_1} \times \dots \times U_{\alpha_N} \rightarrow B$ sending $(h, u_1, \dots, u_N)$ to the product $h u_1 \dots u_N$, is an isomorphism of algebraic varieties.

PROOF. Since B is solvable, one can check that $B = T \times B_u$, where B_u is the subgroup of unipotent elements. We consider the product map

$$f : U_{\alpha_1} \times \dots \times U_{\alpha_N} \rightarrow B_u.$$

This is a T -equivariant map (recall that T acts on the root groups by conjugation), étale at the unit element since the tangent map at e is an isomorphism.

Let $\lambda : \mathbb{G}_m \rightarrow T$ be a *one parameter subgroup* of T , i.e., a group morphism from the multiplicative group to $T \simeq (\mathbb{G}_m)^r$. If $\alpha \in X^*(T)$ is a character, $\alpha \circ \lambda$ is a homomorphism from $\mathbb{G}_m$ to itself, so must be of the form $t \mapsto t^k$ for some integer $k = \langle \alpha, \lambda \rangle$. In particular, λ defines a linear form on $\mathfrak{t}^* \subset \Phi$. Since $\Phi(B)$ is a set of positive roots in Φ , it is contained in some open half-space, so that we can find a one parameter subgroup λ such that $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \Phi(B)$. Then

$$\begin{aligned} \lambda(t) \cdot (x_{\alpha_1}(s), \dots, x_{\alpha_N}(s)) &= (\lambda(t)x_{\alpha_1}(s)\lambda(t)^{-1}, \dots, \lambda(t)x_{\alpha_N}(s)\lambda(t)^{-1}) \\ &= (x_{\alpha_1}(t^{\langle \alpha_1, \lambda \rangle} s), \dots, x_{\alpha_N}(t^{\langle \alpha_N, \lambda \rangle} s)) \end{aligned}$$

converges to e when t goes to zero. We conclude that every point is in the T -orbit of a point at which f is étale, so by equivariance f is étale everywhere. In particular, f is open, and since its image is T -stable, f must be surjective by the same argument as before. Finally, $f^{-1}(e)$ is finite, but also T -stable so consists in fixed points of T . But again for the same reasons, e is the only such fixed point, so f is étale of degree one, hence an isomorphism of algebraic varieties. $\square$

The same kind of arguments can be used to decompose the full group G in terms of the Weyl group, which we can recover at the group level as the quotient by T of its normalizer, $W = N_G(T)/T$.

Theorem 1.3.19. (Bruhat decomposition) *There is a decomposition*

$$G = \coprod_{w \in W} BwB$$

into double cosets, and for each $w \in W$, the product map

$$\left(\prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} U_\alpha \right) \times B \longrightarrow BwB$$

is an isomorphism of algebraic varieties.

Corollary 1.3.20. *The product map $B_u \times B \longrightarrow G$ is an open immersion, and therefore G is rational.*

1.3.7. Classification of homogeneous spaces. Recall that an algebraic variety X is homogeneous if some algebraic group acts transitively on X . The group needs not be affine: an elliptic curve E is a projective algebraic group. More generally, an algebraic group which is a projective variety is called an *abelian variety*.

Over the complex numbers, an abelian variety of dimension n is analytically isomorphic to a quotient $\mathbb{C}^n/\Gamma$, where $\Gamma \simeq \mathbb{Z}^{2n}$ is a lattice.

Theorem 1.3.21. (Borel-Remmert) *A homogeneous projective variety X can be decomposed into the product of an abelian variety A , with a rational homogeneous variety $Y \simeq G/P$, where G is a connected semi-simple affine algebraic group, and P a parabolic subgroup.*

Abelian varieties are extremely interesting objects but are not the subject of these lectures. We will only be concerned with projective varieties that are homogeneous under the action of an affine algebraic group. They are characterized among homogeneous projective varieties by the property of being rational. Their classification is equivalent to the classification of parabolic subgroups, to which we now turn.

Let G denote a connected semi-simple affine algebraic group, B a Borel subgroup, T a maximal torus of G contained in B . From this data we deduce the root system $\Phi \subset \mathfrak{t}^*$. The roots of B are a set of positive roots $\Phi^+ = \Phi(B)$ in Φ , inside which the indecomposable roots form the set Δ of simple roots.

DEFINITION 1.3.22. For any subset I of Δ , we denote by P_I the parabolic group generated by B and the root subgroups $U_{-\alpha}, \alpha \notin I$.

Theorem 1.3.23. *A parabolic subgroup of G is conjugate to one of the P_I 's, and only one of them.*

The Lie algebra $\mathfrak{p}_I$ of P_I will be generated by $\mathfrak{b}$, whose roots we chose to be the positive ones, and the negative root spaces $\mathfrak{g}_{-\alpha}, \alpha \notin I$. We get

$$\mathfrak{p}_I = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(I)} \mathfrak{g}_\alpha,$$

where $\Phi(I)$ denotes the set of roots α such that $\langle \alpha, \omega_i \rangle \geq 0$ for $i \in I$. This Lie algebra naturally splits into the direct sum of the two subalgebras

$$\mathfrak{l}_I = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(I)_0} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I = \bigoplus_{\alpha \in \Phi(I)_+} \mathfrak{g}_\alpha,$$

where $\Phi(I) = \Phi(I)_0 \amalg \Phi(I)_+$ and $\Phi(I)_0$ is defined by the condition that $\langle \alpha, \omega_i \rangle = 0$ for $i \in I$. We know that $\Phi(I)_0$, being a linear section of Φ , is again a root system,

whose Dynkin diagram can be deduced from that of $\mathfrak{g}$ by keeping only the nodes of I . The commutator algebra

$$\mathfrak{h}_I = \mathfrak{t}_I \oplus \bigoplus_{\alpha \in \Phi(I)_0} \mathfrak{g}_\alpha, \quad \mathfrak{t}_I = \bigoplus_{i \notin I} \mathbb{C}H_{\alpha_i},$$

is then semi-simple with this Dynkin diagram, while $\mathfrak{l}_I$ is a direct sum of $\mathfrak{h}_I$ with its center, which is a complement of $\mathfrak{t}_I$ in $\mathfrak{t}$: this is an instance of a *reductive* Lie algebra. Moreover, $\mathfrak{n}_I$ is nilpotent, in the sense that the adjoint action of each of its element is nilpotent. At the group level, we have similarly $P_I = L_I N_I$, where L_I is a reductive subgroup with Lie algebra $\mathfrak{l}_I$, N_I a normal nilpotent subgroup with Lie algebra $\mathfrak{n}_I$, and the intersection of L_I with N_I is finite. This is called the *Levi decomposition*. The group H_I is called the *semi-simple Levi factor*. Note that N_I is uniquely defined, but not L_I and H_I .

REMARK 1.3.24. A reductive Lie algebra can be characterized by the condition that its radical equals its center. At the level of representations, this is equivalent (in characteristic zero) to the fact that any finite-dimensional representation is completely reducible. In practice a nice criterion is the existence of a non-degenerate invariant form (not necessarily the Killing form).

For example, if h is a semisimple element of a semisimple Lie algebra $\mathfrak{g}$, its centralizer $c_{\mathfrak{g}}(h)$ is always reductive (but not semisimple). Indeed, we may suppose that h belongs to our preferred Cartan subalgebra $\mathfrak{t}$, and then

$$c_{\mathfrak{g}}(h) = \mathfrak{t} \oplus \bigoplus_{\alpha(h)=0} \mathfrak{g}_\alpha.$$

The restriction to $c_{\mathfrak{g}}(h)$ of the Killing form of $\mathfrak{g}$ is then easily seen to be non degenerate.

Once we have fixed the Borel subgroup B of G , the P_I 's are the only parabolic subgroups of G containing B . Inclusion defines a natural ordering on this set of parabolic subgroups, from $B = P_\Delta$ to $G = P_\emptyset$. Apart from B , the *minimal parabolic subgroups* are conjugate to the $Q_i = P_{\Delta - \alpha_i}$, and the *maximal parabolic subgroups* are conjugate to the $P_i = P_{\alpha_i}$. Each inclusion induces a projection between rational homogeneous varieties:

$$\begin{array}{ccccc} & & G/B & & \\ & \swarrow & \downarrow & \searrow & \\ G/Q_1 & & G/Q_k & & G/Q_r \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ G/P_{I_1} & & G/P_{I_k} & & G/P_{I_m} \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ G/P_1 & & G/P_k & & G/P_r \end{array}$$

EXAMPLE 1.3.25. For $G = SL_n(\mathbb{C})$, a Borel subgroup is the subgroup B of matrices that are triangular in some fixed basis. Indeed, this is clearly a connected solvable subgroup of G , and we have seen that the quotient $G/B = \mathbb{F}_n$, the variety of complete flags, is projective. This implies that B contains a Borel, and therefore is a Borel subgroup.

A subset $I \subset \Delta$ is a sequence of integers between 1 and $n-1$, and the homogeneous variety G/P_I is the variety of partial flags $\mathbb{F}_I$, the variety of flags of incident

subspaces of $\mathbb{C}^n$ whose dimensions are given by the sequence I . In particular, the G/P_i are the usual Grassmannians. The maps $G/P_I \rightarrow G/P_J$ exist for $J \subset I$, and are defined by forgetting some subspaces.

1.3.8. The Borel-Weil theorem. This famous theorem provides a geometric construction of all the irreducible representations of any simple complex algebraic group G . Consider the complete flag variety G/B parametrizing Borel subgroups of G . Any weight μ of a maximal torus T in B defines a character of B , hence a homogeneous line bundle L_μ on G/B (and in fact any line bundle on G/B is homogeneous). The space of global sections of such a line bundle is naturally a G -module.

Theorem 1.3.26 (Borel-Weil). *If $\mu \in X^*(T)$ is dominant, then*

$$H^0(G/B, L_\mu^*) \simeq V_\mu^*$$

is an irreducible G -module.

A similar statement holds for line bundles on any G/P , if one replaces $X^*(T) \simeq X^*(B)$ by the sublattice $X^*(P)$.

CHAPTER 2

Nilpotent orbits

2.1. The nilpotent cone

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, and $G = \text{Aut}(\mathfrak{g})$ the corresponding adjoint group.

2.1.1. The structure of the nilpotent cone.

DEFINITION 2.1.1. An element $X \in \mathfrak{g}$ is called *nilpotent* (resp. *semi-simple*) if $\text{ad}(X) \in \text{End}(\mathfrak{g})$ is a nilpotent (resp. *semi-simple*) operator.

Basic facts.

- (1) Any element $X \in \mathfrak{g}$ can be decomposed in a unique way as $X = X_s + X_n$ with X_s semi-simple, X_n nilpotent, and $[X_s, X_n] = 0$. This is the *Jordan decomposition*.
- (2) Any nilpotent element $X \in \mathfrak{g}$ can be completed into a $\mathfrak{sl}_2$ -triple (Y, H, X) with H semi-simple, Y nilpotent, and

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

This is the Jacobson-Morozov theorem.

- (3) If X is nilpotent, then it is G -conjugate to tX for any non zero complex number t (one can use conjugation by the one-parameter subgroup generated by H).

Examples of $\mathfrak{sl}_2$ -triples.

- (1) Let α be any positive root. Choose a generator X_α of $\mathfrak{g}_\alpha$ and a generator $X_{-\alpha}$ of $\mathfrak{g}_{-\alpha}$. Let $H_\alpha = [X_\alpha, X_{-\alpha}] \in \mathfrak{h}$. One can check that $\alpha(H_\alpha) \neq 0$. Normalize X_α and $X_{-\alpha}$ in such a way that $\alpha(H_\alpha) = 2$. Then $(X_{-\alpha}, H_\alpha, X_\alpha)$ is a $\mathfrak{sl}_2$ -triple. We have seen that these $\mathfrak{sl}_2$ -triple play an essential role in the study of semisimple Lie algebras.
- (2) Keep the same notations. Let $X = X_{\alpha_1} + \cdots + X_{\alpha_r}$ be a sum of simple root vectors, where r denotes the rank of $\mathfrak{g}$ and $\alpha_1, \dots, \alpha_r$ are the simple roots. One can find $H \in \mathfrak{h}$ such that $\alpha_i(H) = 2$ for all i . Decompose it as $H = a_1 H_{\alpha_1} + \cdots + a_r H_{\alpha_r}$, and let $Y = a_1 X_{-\alpha_1} + \cdots + a_r X_{-\alpha_r}$. Then (Y, H, X) is a $\mathfrak{sl}_2$ -triple.

The last fact above explains the following terminology.

DEFINITION 2.1.2. The *nilpotent cone* $\mathcal{N} \subset \mathfrak{g}$ is the G -invariant set of all nilpotent elements.

In fact $\mathcal{N}$ is an affine algebraic subset of $\mathfrak{g}$, endowed with a natural scheme structure.

Proposition 2.1.3. *The nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is the common zero locus of the G -invariant polynomials on $\mathfrak{g}$ without constant term.*

PROOF. That such a polynomial has to vanish on any nilpotent element $X \in \mathfrak{g}$ follows from the fact that X is G -conjugate to tX for any $t \neq 0$. Conversely, if all the G -invariant polynomials without constant term vanish on an element $X \in \mathfrak{g}$, then the characteristic polynomial of $\text{ad}(X)$ is a monomial. Therefore X is nilpotent. $\square$

What are the G -invariant polynomials on $\mathfrak{g}$? How can we write them down? The most important step towards an answer to these questions is provided by the Chevalley restriction theorem. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset G$ the maximal

torus with Lie algebra $\mathfrak{h}$. Recall that the Weyl group $W = N_G(H)/H \simeq N_G(\mathfrak{h})/H$ acts on $\mathfrak{h}$.

Theorem 2.1.4 (Chevalley's restriction theorem). *The restriction map to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ defines an isomorphism between the algebra of G -invariant polynomials on $\mathfrak{g}$, and the algebra of W -invariant polynomials on $\mathfrak{h}$.*

This is a great step forward because the Weyl group is a *reflection group*. This implies that $\mathbb{C}[\mathfrak{h}]^W$, the algebra of W -invariant polynomials on $\mathfrak{h}$, is a polynomial algebra. In other words, one can find r algebraically independent homogeneous polynomials $f_1, \dots, f_r$, such that

$$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[f_1, \dots, f_r].$$

Let $e_i = \deg(f_i)$ and $d_i = e_i - 1$. One can suppose that $d_1 \leq \dots \leq d_r$. They are basic invariants of $\mathfrak{g}$ called the *fundamental exponents*. The polynomials $f_1, \dots, f_r$ are not uniquely defined, but they can be constructed using finite group theory. Then one should be able to lift them to G -invariant polynomials $F_1, \dots, F_r$ on $\mathfrak{g}$.

It is not difficult to see that the nilpotent cone has codimension r , so it must be cut out set-theoretically by $F_1, \dots, F_r$. More precise results were obtained by Kostant.

Properties (Kostant).

- (1) The nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is the complete intersection of the r invariant hypersurfaces $F_1 = 0, \dots, F_r = 0$.
- (2) The fundamental exponents are symmetric, in the sense that the sum $d_{r+1-i} + d_i$ does not depend on i .
- (3) The sum of the fundamental exponents is equal to the number of positive roots of $\mathfrak{g}$.

Since this is a statement that we will use again and again, we include a

Proof of the Jacobson-Morozov theorem. Let $\mathfrak{c}(X) \subset \mathfrak{g}$ denote the centralizer of X . The invariance property of the Killing form implies that its orthogonal is exactly $[\mathfrak{g}, X]$. Note that if Z belongs to $\mathfrak{c}(X)$, then $\text{ad}(X) \circ \text{ad}(Z)$ is nilpotent hence $K(X, Z) = 0$. Therefore there exists $H \in \mathfrak{g}$ such that $[H, X] = 2X$, and by the properties of the Jordan decomposition this remains true if we replace H by its semisimple part. Otherwise said, we may suppose that H is semisimple.

The relation $[H, X] = 2X$ implies that $\text{ad}(H)$ preserves $\mathfrak{c}(X)$, and we can diagonalize its action. If $[H, U] = tU$ with $t \neq 0$, then $K(H, U) = t^{-1}K(H, [H, U]) = t^{-1}K([H, H], U) = 0$. So there are two cases. Either H is orthogonal to $\mathfrak{c}(X)$, or there exists U such that $[X, U] = [H, U] = 0$ but $K(H, U) \neq 0$.

Let us consider the second case. The relation $K(H, U) \neq 0$ ensures that U is not nilpotent, and using the Jordan decomposition we cannot suppose it is semisimple. Then recall that its centralizer $\mathfrak{c}(U)$ is reductive and contains H and X . Then $X = [H, X]/2$ belongs to $[\mathfrak{c}(U), \mathfrak{c}(U)]$ which is a proper semisimple Lie algebra of $\mathfrak{g}$, and we can use induction.

Now we consider the first case. As we noticed above the orthogonal to $\mathfrak{c}(X)$ is $[X, \mathfrak{g}]$ so we can find $Y \in \mathfrak{g}$ such that $H = [X, Y]$. Let us diagonalize the action of $\text{ad}(H)$ on $\mathfrak{g}$. Since H and X are eigenvectors for the eigenvalues 0 and 2 respectively, the relation $H = [X, Y]$ remains true if we replace Y by its component on the eigenspace corresponding to the eigenvalue -2 . But then $[H, Y] = -2Y$ and we have our $\mathfrak{sl}_2$ -triple. $\square$

2.1.2. Symplectic structures on nilpotent orbits. Let $\mathcal{O} \subset \mathcal{N}$ be a G -orbit. We will see later on that there exist only finitely many such *nilpotent orbits*. Without knowing this we can prove that nilpotent orbits have the very special property of being symplectic manifolds.

First observe that $\mathcal{O}$ being a G -orbit is a smooth locally closed subset of $\mathfrak{g}$. If $X \in \mathcal{O}$, then the tangent space

$$T_X \mathcal{O} = [\mathfrak{g}, X] \simeq \mathfrak{g}/c_{\mathfrak{g}}(X),$$

where $c_{\mathfrak{g}}(X)$ denotes the centralizer of X in $\mathfrak{g}$.

DEFINITION 2.1.5. The Kostant-Kirillov-Souriau skew-symmetric form on $\mathcal{O}$ is defined at the point X by

$$\omega_X(Y, Z) = K(X, [\bar{Y}, \bar{Z}])$$

if $Y = [X, \bar{Y}]$ and $Z = [X, \bar{Z}]$ belong to $T_X \mathcal{O} \simeq [X, \mathfrak{g}]$. By the invariance property of the Killing form, this does not depend on the choice of $\bar{Y}$ and $\bar{Z}$.

Proposition 2.1.6. *The Kostant-Kirillov-Souriau form is a holomorphic symplectic form on $\mathcal{O}$.*

PROOF. We prove that it is non degenerate. Suppose that $Z = [X, \bar{Z}] \in T_X \mathcal{O}$ belongs to the kernel of ω_X . By the invariance of the Killing form, this means that $K(Z, \bar{Y}) = 0$ for any $\bar{Y} \in \mathfrak{g}$. Hence $Z = 0$ since the Killing form is non degenerate. $\square$

Note that we have not used that X is nilpotent. Moreover the Killing form has only been used to identify $\mathfrak{g}$ with its dual. The correct generalization is that any coadjoint orbit is endowed with a natural symplectic form.

Corollary 2.1.7. *Any nilpotent (or coadjoint) orbit has even dimension.*

PROOF. A skew-symmetric form can be non-degenerate only on a vector space of even dimension. $\square$

2.1.3. The Springer resolution. The archetypal symplectic variety is the total space of the cotangent bundle of a smooth variety Z . Indeed, consider the map $p : \Omega_Z \rightarrow Z$. There is a tautological one-form θ on Ω_Z defined by at the point $\Omega \in \Omega_Z$ by $p^* \Omega$. Then the two-form $\omega = d\theta$ is easily seen to be non-degenerate, and is obviously closed.

If $z_1, \dots, z_n$ are local coordinates on Z in a neighbourhood U of some point p , then a one-form can be written over this neighbourhood as $\Omega = y_1 dx_1 + \dots + y_n dx_n$. This yields local coordinates (x, y) in $p^{-1}(U) \subset \Omega_Z$, in which $\omega = dy_1 \wedge dx_1 + \dots + y_n \wedge dx_n$.

Consider the full flag variety $\mathcal{B} = G/B$, where B is a Borel subgroup. As homogeneous vector bundles on $\mathcal{B}$ we have

$$T_{\mathcal{B}} = G \times^B \mathfrak{g}/\mathfrak{b}, \quad \Omega_{\mathcal{B}} = G \times^B \mathfrak{n}.$$

Indeed, the Killing form gives a B -invariant identification between $(\mathfrak{g}/\mathfrak{b})^\vee$ and $\mathfrak{n}$.

Theorem 2.1.8 (Springer's resolution). *The natural projection map*

$$\pi : \Omega_{\mathcal{B}} = G \times^B \mathfrak{n} \rightarrow \mathfrak{g}$$

is a G -equivariant symplectic resolution of singularities of the nilpotent cone.

PROOF. Since $\mathfrak{n}$ and its conjugates are contained in the nilpotent cone, the image of π is contained in $\mathcal{N}$. Conversely, any nilpotent element X defines a one-dimensional solvable Lie algebra in $\mathfrak{g}$, hence it must be contained in a Borel subalgebra. Up to conjugation this Borel subalgebra can be supposed to be $\mathfrak{b}$, and then $X \in \mathfrak{n}$ since the eigenvalues of $\text{ad}(X)$ must be zero. This proves that the image of π is exactly $\mathcal{N}$. In particular $\mathcal{N}$ is irreducible. Moreover π is proper since it factorizes through $G/B \times \mathfrak{g}$ and G/B is complete.

Lemma 2.1.9. *The G -orbit of $X = X_{\alpha_1} + \cdots + X_{\alpha_r}$ is dense in $\mathcal{N}$.*

This orbit is called the *regular orbit* and is denoted $\mathcal{O}_{\text{reg}}$.

PROOF. We have seen how to complete X into a $\mathfrak{sl}_2$ -triple (Y, H, X) by choosing $H \in \mathfrak{h}$ such that $\alpha_i(H) = 2$ for each i . By $\mathfrak{sl}_2$ -theory, the kernel of $\text{ad}(X)$ has its dimension equal to the number of irreducible components of $\mathfrak{g}$ considered as a module over our $\mathfrak{sl}_2$ -triple. Moreover, this number is lower than or equal to (in fact equal to, since all the eigenvalues of H are even) the dimension of the kernel of $\text{ad}(H)$. Since $\alpha(H) \neq 0$ for each root α , this kernel is simply $\mathfrak{h}$ and its dimension is r . This implies that $\text{ad}(X) : \mathfrak{b} \rightarrow \mathfrak{n}$ must be surjective. This means that the B -orbit of X is dense in $\mathfrak{n}$, and therefore the G -orbit of X is dense in $\mathcal{N}$. $\square$

Now we can prove that π is birational. By the proof of the lemma, the source and the target of π have the same dimension. Hence the generic fiber is finite and we want to prove it is a single point. It is enough to prove it for the fiber of our preferred nilpotent element $X = X_{\alpha_1} + \cdots + X_{\alpha_r}$. Since $[H, X] = 2X$ the fiber is preserved by the conjugate action of the one dimensional torus generated by H . Since this group is connected it has to act trivially on each point of the fiber. This means that any conjugate $\mathfrak{n}'$ of $\mathfrak{n}$ containing X is preserved by H . But then the algebra generated by H and $\mathfrak{n}'$ is solvable, hence contained in a Borel subalgebra $\mathfrak{b}'$. Since H is regular its centralizer $\mathfrak{h}$ must be contained in $\mathfrak{b}'$. To conclude, we use the fact that the Borel subalgebras that contain $\mathfrak{h}$ are the conjugates of $\mathfrak{b}$ by the Weyl group, which permutes their sets of simple roots simply transitively. Since $\mathfrak{n}'$ contains X , it contains all the positive simple roots, and therefore it has to coincide with $\mathfrak{n}$.

There remains to prove that π is symplectic. Let us compute the pull-back by π of the Kostant-Kirillov-Souriau form on $\mathcal{O}$. Denote by $\mathfrak{n}_-$ the sum of the negative root spaces in $\mathfrak{g}$, and by $N_- \subset G$ the corresponding connected subgroup. Then it follows from the Bruhat decomposition that $G = N_- B$ and $N_- \cap B = 1$. In particular N_- can be identified with an open subset of G/B over which $\Omega_B = G \times^B \mathfrak{n}$ is a locally trivial fibration, the natural map $N_- \times \mathfrak{n} \rightarrow \Omega_B$ being an isomorphism.

Let $y \in \mathfrak{n}$ belong to the regular orbit. This means that its B -orbit is open in $\mathfrak{n}$, or equivalently that $[\mathfrak{b}, y] = \mathfrak{n}$. A section of the projection map from Ω_B to G/B at (e, y) is given by $u \in N_- \mapsto (u, y)$. Hence a decomposition of $T_{(e, y)} \Omega_B$ into the sum of the vertical tangent space $\mathfrak{n}$ and the tangent space $\mathfrak{n}_-$ of the section. The differential of π at (e, y) is given by $d\pi(u) = u_+ + [y, u_-]$ if $u = (u_+, u_-)$, with $u_+ \in \mathfrak{n}$ and $u_- \in \mathfrak{n}_-$. Therefore the pull-back $\pi^* \omega$ of the Kostant-Kirillov-Souriau form can be computed as

$$\pi^* \omega(u, v) = \omega(d\pi(u), d\pi(v)) = \omega(u_+ + [y, u_-], v_+ + [y, v_-]).$$

Since $[\mathfrak{b}, y] = \mathfrak{n}$ we can write $u_+ = [y, u'_+]$ with $u'_+ \in \mathfrak{b}$. Then by definition of the Kostant-Kirillov-Souriau form we get

$$\omega(u_+, [y, v_-]) = K(y, [u'_+, v_-]) = K([y, u'_+], v_-) = K(u_+, v_-).$$

On the other hand, if we also write $v_+ = [y, v'_+]$ with $v'_+ \in \mathfrak{b}$, we get

$$\omega(u_+, v_+) = \omega([y, u'_+], [y, v'_+]) = K(y, [u'_+, v'_+]) = 0$$

because y and $[u'_+, v'_+]$ belong to $\mathfrak{n}$. Finally, we obtain

$$\pi^*\omega(u, v) = K(u_+, v_-) - K(v_+, u_-) + K(y, [u_-, v_-]).$$

This expression makes perfect sense even if y is not on the regular orbit, so that the pull-back of the Kostant-Kirillov-Souriau form by π extends to a skew-symmetric form which is defined everywhere. Moreover this skew-symmetric form is clearly non-degenerate. With a little more work we can check that $\pi^*\omega$ coincides with the canonical symplectic form on the open subset $p^{-1}(U)$, hence everywhere since this identification is compatible with the left action of G on G/B . $\square$

REMARK 2.1.10. A nilpotent element X belongs to $\mathcal{O}_{reg}$ if and only if its centralizer has dimension r . Indeed, we may suppose that X is contained in the nilradical $\mathfrak{n}$ of our Borel subalgebra $\mathfrak{b}$. If the centralizer of X has dimension r , then the map $ad(X) : \mathfrak{b} \rightarrow \mathfrak{n}$ must be surjective and again this implies that the B -orbit of X is dense in $\mathfrak{n}$. But then, it has to meet the B -orbit of $X_0 = X_{\alpha_1} + \dots + X_{\alpha_r}$, which means that they are conjugate, so that $X \in \mathcal{O}_{reg}$.

Kostant proved that $\mathcal{O}_{reg}$ is exactly the smooth locus of the nilpotent cone $\mathcal{N}$.

It is also worth mentioning that the projectivized nilpotent cone is an interesting projective variety. Recall that given an irreducible projective variety $Z \subset \mathbb{P}V$, one defines its dual variety $Z^* \subset \mathbb{P}V^*$ as the closure of the set of hyperplanes that are tangent to Z at some smooth point. We say that $Z \subset \mathbb{P}V$ is self-dual if it is projectively equivalent to $Z^* \subset \mathbb{P}V^*$. This phenomenon is quite exceptional.

Proposition 2.1.11 (Popov). *The projectivized nilpotent cone $\mathbb{P}\mathcal{N} \subset \mathbb{P}\mathfrak{g}$ is a self-dual projective variety.*

PROOF. We can identify $\mathfrak{g}$ and its dual using the Killing form, hence we consider the dual variety $\mathbb{P}\mathcal{N}^*$ as a subvariety of $\mathbb{P}\mathfrak{g}$, and $\mathcal{N}^* \subset \mathfrak{g}$ its affine cone.

Let $X = \sum_{i \in \Delta} X_{\alpha_i}$. We know that X is regular nilpotent, hence $\mathcal{N} = \overline{G \cdot X}$. The tangent space to $\mathcal{N}$ at X is $[\mathfrak{g}, X]$ and the space of hyperplanes containing it is $[\mathfrak{g}, X]^\perp$. Therefore $\mathcal{N}^*$ is the closure of $G \cdot [\mathfrak{g}, X]^\perp$.

Since $X \in [\mathfrak{g}, X]^\perp$, as follows from the invariance of the Killing form, we deduce that $\mathcal{N} \subset \mathcal{N}^*$. On the other hand, $[\mathfrak{g}, X] \supset \mathfrak{b}$, hence $[\mathfrak{g}, X]^\perp \subset \mathfrak{b}^\perp = \mathfrak{n}$. Since $\mathfrak{n} \subset \mathcal{N}$, we deduce that $\mathcal{N}^* \subset \mathcal{N}$. $\square$

2.1.4. The minimal orbit. Since the adjoint representation is irreducible, its projectivization $\mathbb{P}\mathfrak{g}$ contains a unique closed orbit $X = G/P$. Since it is contained in the closure of every orbit in $\mathbb{P}\mathfrak{g}$, it must be the projectivization of a nilpotent orbit $\mathcal{O}_{min}$, contained in the closure of every non zero nilpotent orbit.

Note that any highest root vector defines a point in X , hence also any long root vector. If we denote the highest root by ψ , we can complete a highest root vector X_ψ into a $\mathfrak{sl}_2$ -triple $(X_{-\psi}, H_\psi, X_\psi)$. The eigenspace decomposition of $ad(H_\psi)$ is

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with $\mathfrak{g}_2 = \mathfrak{g}_\psi$ and $\mathfrak{g}_{-2} = \mathfrak{g}_{-\psi}$. Moreover the tangent space to X at the point x_ψ defined by the line $\mathfrak{g}_\psi$ is

$$T_{x_\psi}X = [\mathfrak{g}_{-\psi}, \mathfrak{g}_\psi] \oplus \mathfrak{g}_1.$$

In particular the dimension of X is one plus the number of positive roots α such that $\alpha(H_\psi) = 1$.

2.2. Classification of nilpotent orbits

2.2.1. The number of nilpotent orbits is finite. We have seen that any nilpotent element can be completed into a $\mathfrak{sl}_2$ -triple. To what extent do these triples depend on their components? As much as they can, that is:

- (1) (Kostant) Any two $\mathfrak{sl}_2$ -triples with the same positive nilpotent element are conjugate.
- (2) (Malcev) Any two $\mathfrak{sl}_2$ -triples with the same semi-simple element are conjugate.

This implies that we can encode any nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ by the orbit of the semi-simple elements that can occur in the corresponding $\mathfrak{sl}_2$ -triples. This is quite convenient because semi-simple orbits are much simpler to understand.

Proposition 2.2.1. *The set of semi-simple orbits is in bijection with $\mathfrak{h}/W$.*

In more sophisticated terms, the fact is that the only closed orbits in $\mathfrak{g}$ are the semi-simple ones. The claim is then a reformulation of Chevalley's restriction theorem, restated as $\mathfrak{g}/G \simeq \mathfrak{h}/W$.

PROOF. Any semi-simple element is contained in a Cartan subalgebra, and all Cartan subalgebras are conjugate, hence any semi-simple orbit meets our given $\mathfrak{h}$. There remains to show that H and H' in $\mathfrak{h}$ are G -conjugate if and only if they are W -conjugate. Suppose that $H = \text{Ad}(g)H'$. Then $\mathfrak{h}$ and $\text{Ad}(g)\mathfrak{h}$ are two Cartan subalgebras containing H , hence contained in $c_{\mathfrak{g}}(H)$. But this is a reductive algebra, inside which $\mathfrak{h}$ and $\text{Ad}(g)\mathfrak{h}$ are certainly two Cartan subalgebras. Hence they must be conjugate by an element k in the adjoint group of $c_{\mathfrak{g}}(H)$. Then $\text{Ad}(kg) = \text{Ad}(k)\text{Ad}(g)\mathfrak{h} = \mathfrak{h}$ and $\text{Ad}(k)\text{Ad}(g)H' = \text{Ad}(k)H = H$. Thus kg defines an element of W mapping H' to H . $\square$

If X is nilpotent and we complete it into a $\mathfrak{sl}_2$ -triple (Y, H, X) , we can suppose that $H \in \mathfrak{h}$. Observe that by $\mathfrak{sl}_2$ -theory, $\alpha(H) \in \mathbb{Z}$ for any root α . Moreover, using the action of the Weyl group we may suppose that $\alpha(H) \geq 0$ for any positive root α . The orbit of H is then completely characterized by the set of non negative integers $\alpha_1(H), \dots, \alpha_r(H)$.

Proposition 2.2.2. *One has $\alpha_i(H) \in \{0, 1, 2\}$ for all i .*

PROOF. By $\mathfrak{sl}_2$ -theory, since X_{α_i} is an eigenvector of $\text{ad}(H)$ for the eigenvalue $\alpha_i(H)$, we know that $[Y, X_{\alpha_i}]$ is an eigenvector of $\text{ad}(H)$ for the eigenvalue $\alpha_i(H) - 2$. But Y must be a combination of negative root vectors, and therefore $[Y, X_{\alpha_i}]$ must belong to $\mathfrak{b}_-$ since we can never obtain a positive root by adding a simple root to a negative one. So the corresponding eigenvalue cannot be positive, that is, $\alpha_i(H) - 2 \leq 0$ as claimed. But beware that it could also happen that $[Y, X_{\alpha_i}] = 0$. In that case X_{α_i} is a lowest weight vector for our $\mathfrak{sl}_2$ -triple, and therefore the eigenvalue of $\text{ad}(H)$ must be non positive, that is $\alpha_i(H) \leq 0$, hence in fact $\alpha_i(H) = 0$. $\square$

As an immediate consequence, we obtain the important result:

Theorem 2.2.3. *There exist only finitely many nilpotent orbits.*

Corollary 2.2.4. *The nilpotent cone is normal.*

PROOF. The singular locus of $\mathcal{N}$ is a union of finitely many nilpotent orbits, all of codimension at least two since they all have even dimensions. Therefore $\mathcal{N}$ is smooth in codimension one. Being a complete intersection, it is normal by Serre's criterion. $\square$

REMARK 2.2.5. Beware that a nilpotent orbit closure is not always normal, and that deciding whether it is or not is a difficult problem, still not completely solved for the exceptional Lie algebras.

The theorem also provides us with a concrete way to encode each nilpotent orbit, by a *weighted Dynkin diagram*: just give the weight $\alpha_i(H) \in \{0, 1, 2\}$ to the node corresponding to the simple root α_i . For example the weighted Dynkin diagram corresponding to the regular orbit $\mathcal{O}_{reg}$ has all its vertices weighted by 2.

EXAMPLE 2.2.6. Nilpotent orbits in $\mathfrak{sl}_n$ are classified by their Jordan type, which is encoded by a partition of n . We will denote by $\mathcal{O}_\lambda$ the nilpotent orbit corresponding to the partition λ . The regular orbit $\mathcal{O}_{reg} = \mathcal{O}_n$ (a unique Jordan block), while the minimal orbit $\mathcal{O}_{min} = \mathcal{O}_{21 \dots 1}$. The regular orbit is represented by the matrix X whose non-zero entries are ones above the diagonal. We complete it into a sl_2 -triple (Y, H, X) with

$$H = \begin{pmatrix} h_1 & 0 & 0 & \cdots & & \\ 0 & h_2 & 0 & \cdots & & \\ 0 & 0 & h_3 & \cdots & & \\ & & & \cdots & h_{n-1} & 0 \\ & & & & 0 & h_n \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ y_1 & 0 & 0 & \cdots & & \\ 0 & y_2 & 0 & \cdots & & \\ & & & \cdots & 0 & 0 \\ & & & & \cdots & y_{n-1} & 0 \end{pmatrix},$$

where $h_i = n+1-2i$ and $y_i = h_1 + \cdots + h_i$. One can deduce a sl_2 -triple corresponding to any nilpotent orbit by treating each Jordan block separately. The only point is that the semisimple element H that one obtains this way does not verify the normalization condition $\alpha_i(H) = h_i - h_{i+1} \geq 0$. This is easily corrected simply by putting the diagonal entries of H in non increasing order. One can then read off the resulting matrix the weighted Dynkin diagram of the orbit.

For $\mathfrak{sl}_6$ we get ten non zero orbits, whose weighted Dynkin diagrams are the following:

$\mathcal{O}_6$	$\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$	$\mathcal{O}_{321}$	$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$
$\mathcal{O}_{51}$	$\begin{array}{ccccc} 2 & 2 & 0 & 2 & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$	$\mathcal{O}_{2^3}$	$\begin{array}{ccccc} 0 & 0 & 2 & 0 & 0 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$
$\mathcal{O}_{42}$	$\begin{array}{ccccc} 2 & 0 & 2 & 0 & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$	$\mathcal{O}_{31^3}$	$\begin{array}{ccccc} 2 & 0 & 0 & 0 & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$
$\mathcal{O}_{33}$	$\begin{array}{ccccc} 0 & 2 & 0 & 2 & 0 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$	$\mathcal{O}_{2^2 1^2}$	$\begin{array}{ccccc} 0 & 1 & 0 & 1 & 0 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$
$\mathcal{O}_{41^2}$	$\begin{array}{ccccc} 2 & 1 & 0 & 1 & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$	$\mathcal{O}_{21^4}$	$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$

The major drawback of weighted Dynkin diagrams is that it is not clear how to characterize which, among the 3^n possible weighted Dynkin diagrams, really

represent nilpotent orbits. This is why other, more effective methods have been developed, notably Bala-Carter theory that will be explained below.

2.2.2. Richardson orbits. The Springer resolution has shown how to obtain the nilpotent cone by collapsing a vector bundle over G/B . One can wonder if other orbit closures can be obtained from other G -homogeneous spaces G/P .

Consider a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, and its Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Since $\mathfrak{u}$ is contained in the nilpotent cone, and since there are only finitely many nilpotent orbits, there exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{p}}$ meeting it along a dense subset. Such an orbit is a *Richardson orbit*. For example the regular orbit is Richardson.

Proposition 2.2.7 (Richardson). *Let $\mathfrak{p}$ be a parabolic subalgebra, $\mathcal{O}_{\mathfrak{p}}$ the associated Richardson orbit and $P \subset G$ the corresponding parabolic subgroup. Then the natural map*

$$\pi_{\mathfrak{p}} : \Omega_{G/P} = G \times^P \mathfrak{u} \rightarrow \mathfrak{g}$$

is a generically finite G -equivariant map over its image $\bar{\mathcal{O}}_{\mathfrak{p}}$.

EXAMPLE 2.2.8. A parabolic subgroup P of SL_n is the stabilizer of a flag $V_{\bullet}$ of subspaces $0 \subset V_1 \subset \cdots \subset V_m \subset \mathbb{C}^n$ of prescribed dimensions. The cotangent bundle of G/P can be identified with the space of pairs $(V_{\bullet}, x)$ where $x \in \mathfrak{sl}_n$ is such that $x(V_i) \subset V_{i-1}$ for all i (with the usual convention that $V_0 = 0$ and $V_{m+1} = \mathbb{C}^n$). We say that the flag $V_{\bullet}$ is adapted to x .

A flag that is canonically attached and adapted to x is the flag defined by $V_i = \text{Ker}(x^i)$. Clearly it is the only flag adapted to x with the same dimensions. If $\mathfrak{p}$ is the parabolic subalgebra defined by this flag, this implies that $\pi_{\mathfrak{p}} : \Omega_{SL_n/P} \rightarrow \mathfrak{sl}_n$ is birational, hence a resolution of singularities of $\bar{\mathcal{O}}_{\mathfrak{p}}$. Moreover any nilpotent orbit in $\mathfrak{sl}_n$ is a Richardson orbit.

Beware that the degree of $\pi_{\mathfrak{p}}$ is not always equal to one. A generically finite proper and surjective morphism from a smooth variety to a singular one is sometimes called an *alteration of singularities*.

EXAMPLE 2.2.9. The projective space $\mathbb{P}^{2n-1}$ is acted on transitively by SL_{2n} , but also by the symplectic group Sp_{2n} . In the symplectic setting a line $L \subset \mathbb{C}^{2n}$ should be viewed as corresponding to the parabolic subgroup of Sp_{2n} stabilizing the symplectic flag $0 \subset L \subset L^{\perp} \subset \mathbb{C}^{2n}$. Moreover $\Omega_{\mathbb{P}^{2n-1}}$ is identified with the space of pairs (L, x) with $x \in \mathfrak{sp}_{2n}$ adapted to the symplectic flag attached to L .

We claim that the projection map $\Omega_{\mathbb{P}^{2n-1}} \rightarrow \mathfrak{sp}_{2n}$ is not birational over its image $\bar{\mathcal{O}}_{\mathfrak{p}}$. This can be seen as follows. We know that the projection map $\Omega_{\mathbb{P}^{2n-1}} \rightarrow \mathfrak{sl}_{2n}$ is birational over its image $\bar{\mathcal{O}}$, where $\mathcal{O}$ is the minimal nilpotent orbit in $\mathfrak{sl}_{2n}$. Denote by J the skew-symmetric matrix defining the symplectic form. One can check that the natural map

$$p : \mathfrak{sl}_{2n} \rightarrow \mathfrak{sp}_{2n}, \quad y \mapsto y - J^{-1}y^t J$$

sends $\bar{\mathcal{O}}$ to $\bar{\mathcal{O}}_{\mathfrak{p}}$. Since $p(y) = p(-J^{-1}y^t J)$, the induced map $p : \bar{\mathcal{O}} \rightarrow \bar{\mathcal{O}}_{\mathfrak{p}}$ has degree at least two – and in fact degree two. This implies that $\Omega_{\mathbb{P}^{2n-1}} \rightarrow \mathfrak{sp}_{2n}$ has degree two over its image $\bar{\mathcal{O}}_{\mathfrak{p}}$.

2.2.3. Resolutions of singularities. A variant of the preceding construction allows to define a canonical resolution of singularities for any nilpotent orbit closure. We start with a nilpotent element e in some nilpotent $\mathcal{O}$. We complete it into a $\mathfrak{sl}_2$ -triple (e, h, f) . Then h defines a grading on $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, and we denote by $\mathfrak{p}$ the parabolic subalgebra $\mathfrak{g}_{\geq 0}$, with obvious notations.

Lemma 2.2.10. *The parabolic algebra $\mathfrak{p}$ only depends on e , and not on h .*

Let P denote the parabolic subgroup of the adjoint group G , with Lie algebra $\mathfrak{p}$. Each subspace $\mathfrak{g}_{\geq i}$ of $\mathfrak{g}$ is a P -module. Consider the natural map

$$\varphi : G \times^P \mathfrak{g}_{\geq 2} \rightarrow \mathfrak{g}.$$

We know that $e \in \mathfrak{g}_{\geq 2}$ and that the map $Ad(e) : \mathfrak{p} \rightarrow \mathfrak{n}_2$ is surjective, so that the P -orbit of e is dense in $\mathfrak{g}_2$. This implies that the image of φ is exactly the closure of $\mathcal{O}$. Moreover, by the previous lemma the preimage of e is a unique point, hence φ is birational. We have proved:

Theorem 2.2.11. *The map $\varphi : G \times^P \mathfrak{g}_{\geq 2} \rightarrow \overline{\mathcal{O}} \subset \mathfrak{g}$ is a G -equivariant resolution of singularities.*

Using the same argument as for the Springer resolution of the nilpotent cone, one can check that the Kostant-Kirillov-Souriau symplectic form on $\mathcal{O}$ extends to a global skew-symmetric form on $\varphi : G \times^P \mathfrak{g}_{\geq 2}$. This exactly means that:

Corollary 2.2.12. *Every nilpotent orbit closure is a symplectic variety.*

Beware that the extended skew-symmetric form will in general not remain non-degenerate on the boundary of the nilpotent orbit. When it does, the orbit closure is said to have *symplectic singularities*. Such singularities are quite rare. The closure of a Richardson orbit does admit symplectic singularities when its Springer type alteration of singularities has degree one. For example this is always the case for orbits in $\mathfrak{sl}_n$. It will also be true for even orbits, that we define as follows.

DEFINITION 2.2.13. The orbit $\mathcal{O}$ is *even* if $\mathfrak{g}_{\geq 1} = \mathfrak{g}_{\geq 2}$. Equivalently the weighted Dynkin diagram of $\mathcal{O}$ has weights 0 or 2 only.

Since $\mathfrak{g}_{\geq 1}$ is the radical of $\mathfrak{p}$ (that we denoted by $\mathfrak{u}$ in the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$), we get in this case that $\pi_{\mathfrak{p}} = \varphi$ is birational. We conclude:

Proposition 2.2.14. *Every even orbit is Richardson and its closure admits a Springer type resolution of singularities, which is symplectic. In particular an even orbit closure has symplectic singularities.*

REMARK 2.2.15. Note that the normalization $\tilde{\mathcal{O}}$ of $\overline{\mathcal{O}}$ is Gorenstein with trivial canonical bundle. Indeed the mere existence of the Kostant-Kirillov-Souriau form implies that the canonical bundle is trivial on the smooth locus, and the singular locus has codimension at least two since every nilpotent orbit has even dimension. When it is a symplectic resolution of singularities, the birational morphism φ is thus crepant. In general, the fact that the pull-back by φ of the Kostant-Kirillov-Souriau form extends to a global two-form implies that the push-forward by φ of the canonical bundle is trivial. By a result of Flenner this ensures that $\tilde{\mathcal{O}}$ always has rational singularities.

Finally, let us mention that by a relatively recent result of Kaledin, the singular locus of $\overline{\mathcal{O}}$ is exactly the complement of $\mathcal{O}$. Its codimension is at least two, but it can be bigger.

One can prove that a nilpotent orbit closure admitting a symplectic resolution must be Richardson. Moreover any symplectic resolution of a Richardson orbit is a Springer type alteration $\pi_{\mathfrak{p}}$ [10], and this happens precisely when $\pi_{\mathfrak{p}}$ has degree one.

2.2.4. Mukai's flops. An interesting point is that two non conjugate parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ may define the same Richardson orbit.

Proposition 2.2.16. *Suppose that two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ have conjugate Levi subalgebras. Then they define the same Richardson orbit.*

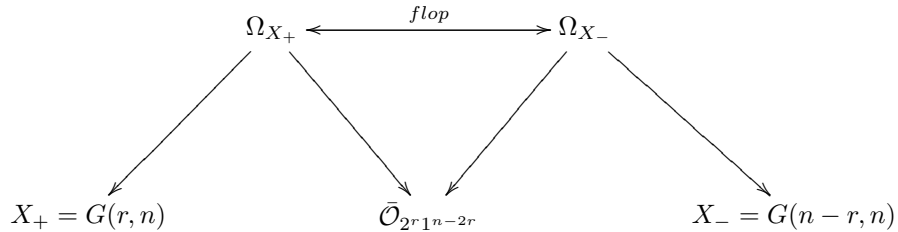
PROOF. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition. The closure of the corresponding Richardson orbit is $G \cdot \mathfrak{u}$. Let $\mathfrak{z}$ be the center of $\mathfrak{l}$. Let us suppose that $\mathfrak{p}$ is standard, in which case $\mathfrak{z} \subset \mathfrak{h}$ is contained in the Cartan subalgebra and $\mathfrak{u} \subset \mathfrak{n}$. For any root space $\mathfrak{g}_{\alpha}$ contained in $\mathfrak{u}$ one can find $Z \in \mathfrak{z}$ such that $\alpha(Z) \neq 0$, therefore $[\mathfrak{z}, \mathfrak{g}]$ contains $\mathfrak{u}$. But then the closure of $G \cdot \mathfrak{z}$ contains $G \cdot (\mathfrak{z} + \mathfrak{u})$, which is closed, hence they are equal. This implies that

$$G \cdot \mathfrak{u} = \overline{G \cdot \mathfrak{z}} \cap \mathcal{N}.$$

The proof is now complete, since we have described the Richardson orbit corresponding to $\mathfrak{p}$ in terms of its Levi subalgebra $\mathfrak{l}$ only. $\square$

For example, a nilpotent orbit closure in type A can have an arbitrary large number of different symplectic resolutions. Indeed, the type of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{gl}_n$ is prescribed by the dimensions of the spaces in the flag it preserves. This can be any increasing sequence $d_0 = 0, d_1, \dots, d_k = n$. The Levi subalgebra of $\mathfrak{p}$ is then isomorphic with the sum of the $\mathfrak{gl}_{m_i}$ for $m_i = d_i - d_{i-1}$, $1 \leq i \leq k$. We have $d_i = m_1 + \dots + m_i$, but if we permute the m_i 's arbitrarily we can get a different increasing sequence $d'_0 = 0, d'_1, \dots, d'_k = n$, hence another type of parabolic subalgebra defining the same Richardson orbit. Moreover the corresponding Springer type resolution of singularities of the orbit closure will be different, although they can be proved to be *deformation equivalent*.

A typical example is *Mukai's flop* between two distinct resolutions of the closure of an orbit of type $\mathcal{O}_{2r, 1^{n-2r}}$ in $\mathfrak{sl}_n$. This is the orbit of matrices of rank r and square zero. The image and the kernel provide us with maps to two different Grassmannians. Passing to the closure we get the following picture:



More explicitly, note that $\Omega_{X_{\pm}}$ is the total space of $\text{Hom}(Q, S)$ on the Grassmannian, where Q and S are the quotient and tautological bundle. This is the subspace in $\text{Hom}(V, S)$ of morphisms vanishing on S , hence the natural map to $\mathfrak{sl}_n$ whose image is our orbit closure.

2.2.5. Bala-Carter theory.

Associated parabolic subalgebras. Recall that if (Y, H, X) is a $\mathfrak{sl}_2$ -triple, the semi-simple element H has integer eigenvalues. The decomposition of $\mathfrak{g}$ into eigenspaces,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

is a *grading* of $\mathfrak{g}$, in the sense that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, as follows from the Jacobi identity. In particular $\mathfrak{g}_0$, $\mathfrak{g}_{\geq 0}$ and $\mathfrak{g}_{>0}$, with obvious notations, are graded subalgebras of $\mathfrak{g}$. We have seen that we may suppose that $H \in \mathfrak{h}$, in which case each $\mathfrak{g}_i$, for $i \neq 0$, is just a sum of root spaces. Moreover we may suppose that $\alpha(H) \geq 0$ for any positive root α . This implies that $\mathfrak{g}_{\geq 0} \supset \mathfrak{b}$ is a parabolic subalgebra with radical $\mathfrak{g}_{>0}$ and Levi subalgebra $\mathfrak{g}_0 = c_{\mathfrak{g}}(H)$, a reductive subalgebra containing $\mathfrak{h}$.

Beware that this parabolic subalgebra does not characterize the orbit of X : two weighted Dynkin diagrams with the same zeroes will define the same conjugacy class of parabolic subalgebras.

Distinguished orbits. The main idea of Bala-Carter theory is to use induction through proper Levi subalgebras. Indeed, if a nilpotent element is contained in a proper Levi subalgebra $\mathfrak{l}$, then it is also contained in its semi-simple part $[\mathfrak{l}, \mathfrak{l}]$, which has smaller rank than $\mathfrak{g}$. Also recall that $\mathfrak{l}$ has a non trivial center $\mathfrak{z}$, consisting in semi-simple elements, and that $\mathfrak{l}$ and $\mathfrak{z}$ are the centralizers in $\mathfrak{g}$ one of each other.

A consequence is that if $X \in \mathfrak{l} \neq \mathfrak{g}$, then $c_{\mathfrak{g}}(X)$ contains a non zero semi-simple element. This element must be contained in $\mathfrak{g}_0$ and therefore $\dim \mathfrak{g}_2 < \dim \mathfrak{g}_0$.

Conversely, suppose that $\dim \mathfrak{g}_2 < \dim \mathfrak{g}_0$. Then the centralizer $\mathfrak{k}$ of the full $\mathfrak{sl}_2$ -triple is non zero. One can check that $\mathfrak{k}$ is reductive, hence it contains some non zero semi-simple element H . Then the centralizer of H is a proper reductive subalgebra of $\mathfrak{g}$ containing X .

DEFINITION 2.2.17. A nilpotent element $X \in \mathfrak{g}$ is distinguished if it is not contained in any proper Levi subalgebra.

With the same notations as before, we have proved that X is distinguished if and only if $\dim \mathfrak{g}_2 = \dim \mathfrak{g}_0$. Independently of the grading, this can be rewritten as follows: we have the parabolic algebra $\mathfrak{p} = \mathfrak{g}_{\geq 0}$, with its nilpotent radical $\mathfrak{u} = \mathfrak{g}_{>0}$ and its Levi part $\mathfrak{l} = \mathfrak{g}_0$. Then the condition can be rewritten as

$$\dim \mathfrak{l} = \dim \mathfrak{u} / [\mathfrak{u}, \mathfrak{u}].$$

A parabolic subalgebra satisfying this condition will be called *distinguished*. Note that the dimension on the right hand side is in fact easy to compute in terms of roots: it is the number of *indecomposable roots* in $\mathfrak{u}$, that is, roots contributing to $\mathfrak{u}$ which cannot be written as the sum of two such roots.

In order to complete the induction, there remains to prove that each distinguished parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ in $\mathfrak{g}$ comes from a distinguished nilpotent element. Let $\mathcal{O}$ be the unique nilpotent orbit meeting $\mathfrak{u}$ along a dense subset, and Z be a point in this intersection. As before we know that $[\mathfrak{p}, Z] = \mathfrak{u}$. Let $\mathfrak{a}$ (resp. $\mathfrak{a}_-$) denote the sum of the root spaces in $\mathfrak{u}$ corresponding to indecomposable roots (resp. to their opposite roots). Then $\mathfrak{a}$ is a complement to $[\mathfrak{u}, \mathfrak{u}]$ in $\mathfrak{u}$, and $[\mathfrak{a}, \mathfrak{a}_-] = \mathfrak{l}$. Recall that the hypothesis that $\mathfrak{p}$ is distinguished means that $\mathfrak{a}$ and $\mathfrak{l}$ have the same dimension.

Write $Z = U + U'$ with $U \in \mathfrak{a}$ and $U' \in [\mathfrak{u}, \mathfrak{u}]$. From $[\mathfrak{p}, Z] = \mathfrak{u}$ we deduce that necessarily, $[\mathfrak{l}, U] = \mathfrak{a}$. In particular $[\mathfrak{g}, U] \supset \mathfrak{a}$, and by taking orthogonals with respect to the Killing form we get that $c_{\mathfrak{g}}(U) \subset \mathfrak{a}^\perp$, hence $c_{\mathfrak{g}}(U) \cap \mathfrak{a}_- \subset \mathfrak{a}_- \cap \mathfrak{a}^\perp = 0$. We deduce that $[\mathfrak{a}_-, U]$ has the same dimension as $\mathfrak{a}_-$, hence $\mathfrak{l}$, so that they must be equal.

Now we choose $H \in \mathfrak{h}$ such that $\alpha_i(H) = 0$ if the simple root α_i is a root of $\mathfrak{l}$, and $\alpha_i(H) = 2$ otherwise. Then $\alpha(H) = 2$ for any root α contributing to $\mathfrak{a}$, hence $[H, U] = 2U$. Moreover, since $[\mathfrak{a}_-, U] = \mathfrak{l}$ there exists $V \in \mathfrak{a}_-$ such that $[U, V] = H$. Finally U is a distinguished nilpotent element in $\mathfrak{g}$ and $\mathfrak{p}$ is the associated parabolic subalgebra. We have proved:

Theorem 2.2.18. *There exists a bijection between nilpotent orbits in $\mathfrak{g}$ and G -conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p})$, with $\mathfrak{l}$ a Levi subalgebra in $\mathfrak{g}$ and $\mathfrak{p}$ a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$.*

Note that the previous proof also constructs for us a representative of the orbit $\mathcal{O}$ corresponding to $(\mathfrak{l}, \mathfrak{p})$. Indeed, we just need to decompose $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ and then $\mathfrak{u} = \mathfrak{a} \oplus [\mathfrak{u}, \mathfrak{u}]$, and we know that a generic element of $\mathfrak{a}$ has to belong to $\mathcal{O}$.

EXAMPLE 2.2.19. If $\mathfrak{g} = \mathfrak{sl}_n$, a parabolic standard subalgebra has a Levi subalgebra consisting in block matrices of size $l_1, \dots, l_k$ with $l_1 + \dots + l_k = n$. Then $\dim \mathfrak{l} = l_1^2 + \dots + l_k^2$ while $\dim \mathfrak{a} = l_1 l_2 + \dots + l_{k-1} l_k$. One deduces that no proper parabolic subalgebra is distinguished. More generally the Levi subalgebras of $\mathfrak{g}$ do not contain any proper parabolic subalgebra. Therefore there is exactly one nilpotent orbit for each class of Levi subalgebra, hence for each partition of n .

EXAMPLE 2.2.20. We classify the nilpotent orbits in $\mathfrak{g}_2$. Let α_1, α_2 be the two simple roots, the first one being short. Up to conjugation there are four parabolic algebras $\mathfrak{b}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{g}_2$ and four Levi subalgebras $\mathfrak{h}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{g}_2$ with respective semisimple parts $0, \mathfrak{sl}_2, \mathfrak{sl}_2, \mathfrak{g}_2$. The first three account for three orbits, those of $0, X_{\alpha_1}, X_{\alpha_2}$. There remains to classify distinguished parabolic subalgebras of $\mathfrak{g}_2$. As always $\mathfrak{b}$ is distinguished but $\mathfrak{g}_2$ is not. The positive roots in $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) are $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$ (resp. $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$), among which the indecomposable ones are α_1 and $\alpha_1 + \alpha_2$ (resp. $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$). Since $\mathfrak{l}_1$ and $\mathfrak{l}_2$ have dimension four, we see that $\mathfrak{p}_2$, which gives four indecomposable roots, is distinguished, while $\mathfrak{p}_1$ is not.

2.2.6. The subregular orbit. Let us come back to the Springer resolution $\pi : \Omega_{\mathcal{B}} = G \times^B \mathfrak{n} \rightarrow \mathcal{N}$. We know it is a isomorphism over the regular orbit $\mathcal{O}_{reg}$. More precisely, we have seen that the regular orbit is the orbit of the sum of the simple root vectors

$$X = X_{\alpha_1} + \dots + X_{\alpha_r},$$

to which we can add an arbitrary combination of the root vectors corresponding to non simple positive roots. Since the simple root vectors had been chosen arbitrarily (but non zero, of course), any linear combination

$$X' = t_{\alpha_1} X_{\alpha_1} + \dots + t_{\alpha_r} X_{\alpha_r}$$

with $t_{\alpha_1} \dots t_{\alpha_r} \neq 0$ is also in the regular orbit. But if one of the coefficient is taken to be zero, then the centralizer of X' has dimension $r + 2$ and semisimple part isomorphic to $\mathfrak{sl}_2$. This shows that the pull-back by π of the complement of the regular orbit is $G \times^B \mathfrak{t}$ where $\mathfrak{t} \subset \mathfrak{n}$ is a union of hyperplanes indexed by the simple

roots. Moreover its image is the closure of finitely many orbits of codimension two in $\mathcal{N}$. Since two simple roots are conjugate under the Weyl group when they have the same length, two hyperplanes corresponding to two roots of the same length will yield the same orbit. It is not a priori clear that the same conclusion should hold for two simple roots α and α' of different lengths, but we already know that it is true when $\mathfrak{g}$ has rank two: indeed we have seen that for $\mathfrak{sp}_4$ and $\mathfrak{g}_2$, there is only one nilpotent orbit of codimension two. But then the general case follows: indeed we can suppose that α and α' define adjacent vertices in the Dynkin diagram; then we can consider them inside the simple Lie algebra they generate, which has rank two. And the conclusion goes through. We have proved:

Proposition 2.2.21. *The complement in $\mathcal{N}$ of the regular orbit $\mathcal{O}_{reg}$ in the closure of a unique orbit of codimension two.*

This orbit is called the *subregular orbit* and denoted $\mathcal{O}_{subreg}$. It is particularly important because the singular locus of $\mathcal{N}$ is exactly the closure of $\mathcal{O}_{subreg}$. Moreover, taking a transverse slice to $\mathcal{O}_{subreg}$ at some point yields an affine surface with an isolated singularity. If $\mathfrak{g}$ is simply laced, it has been proved by Grothendieck that this is an *ADE* singularity of the same type as $\mathfrak{g}$ [31].

2.2.7. The closure ordering. There is a natural ordering on the set of nilpotent orbits, the closure ordering. We simply let $\mathcal{O} \geq \mathcal{O}'$ if the Zariski closure $\overline{\mathcal{O}}$ contains $\mathcal{O}'$.

The closure ordering has been determined for all the simple Lie algebras. One of the most efficient tools is to use the desingularization $G \times^P \mathfrak{u}$ of the orbit closure $\overline{\mathcal{O}}$ that we constructed below. It is clear that $\mathcal{O} \geq \mathcal{O}'$ if and only if $\mathcal{O}'$ meets the linear space $\mathfrak{u}$. We can for example deduce the following statement.

Proposition 2.2.22 (Gerstenhaber). *Let λ be a partition of n , and $\mathcal{O}_\lambda \subset \mathfrak{sl}_n$ the nilpotent orbit of matrices whose Jordan blocks have for sizes the parts λ_i of λ . Then $\mathcal{O}_\lambda \geq \mathcal{O}_\mu$ if and only if*

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \forall i.$$

This condition defines a partial order on the set of partitions of a same integer, that we denote by $\lambda \geq \mu$. This is called the *dominance ordering*.

PROOF. Recall that x belongs to $\mathcal{O}_\lambda$ if and only if the kernel of x^k has dimension $\lambda_1^* + \cdots + \lambda_k^*$ for each k , where λ^* denotes the dual partition of λ . Since the dimension of kernels can only increase when we go to the closure, the condition that $\mathcal{O}_\lambda \geq \mathcal{O}_\mu$ implies that $\lambda^* \leq \mu^*$ for the dominance ordering. So it implies that $\lambda \geq \mu$ by the following lemma:

Lemma 2.2.23. *The dominance ordering is reversed by duality: $\lambda \geq \mu$ if and only if $\lambda^* \leq \mu^*$.*

PROOF. Write $\lambda \mapsto \mu$ if $\lambda > \mu$ and there is no partition ν such that $\lambda > \nu > \mu$. Then there exists a sequence of parts of λ of the form $(p+1, p^a, q)$, with $p > q$ and $a \geq 0$, such that the corresponding sequence of parts in μ is $(p^{a+1}, q+1)$ – all the other parts being the same in λ and μ . This implies that $\lambda \mapsto \mu$ if and only if $\mu^* \mapsto \lambda^*$, hence the claim. $\square$

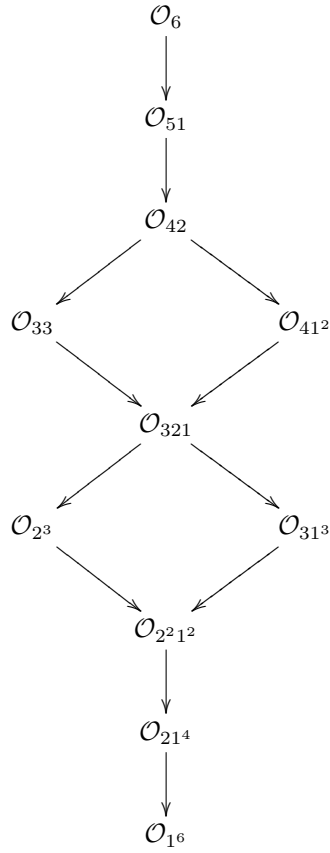
Conversely, suppose that $\lambda \mapsto \mu$. We need to prove that the closure of $\mathcal{O}_\lambda$ contains $\mathcal{O}_\mu$. It is enough to check this for $\lambda \mapsto \mu$, and by the previous lemma it suffices to show that the closure of $\mathcal{O}_{p+1,q}$ contains $\mathcal{O}_{p,q+1}$. Consider x with two

Jordan blocks of size $p + 1$ and q , that is $x = \sum_{i \neq p+2} e_i^* \otimes e_{i-1}$. We can easily complete it into a $\mathfrak{sl}_2$ -triple by treating separately the two blocks, hence reducing to the regular orbits in $\mathfrak{sl}_{p+1}$ and $\mathfrak{sl}_q$. In particular we get a semisimple element h in diagonal form, to which we associate the linear space $\mathfrak{u} \subset \mathfrak{sl}_{p+q}$. Then the element

$$x' = \sum_{i=3}^{p+1} e_i^* \otimes e_{i-1} + e_{p+2}^* \otimes e_1 + \sum_{j=p+3}^{p+q+1} e_j^* \otimes e_{j-1}$$

belongs to $\mathfrak{u}$ and has Jordan type $(p, q + 1)$. This completes the proof. $\square$

EXAMPLE 2.2.24. Recall that we have listed ten non trivial nilpotent orbits in $\mathfrak{sl}_6$. The closure ordering is the following:



CHAPTER 3

Prehomogeneous spaces, generalities

3.1. Prehomogeneous vector spaces

In this section we discuss prehomogeneous spaces, vector spaces on which a Lie group acts almost transitively. More than the fact that they are the natural affine generalizations of the homogeneous spaces, our motivation is that a large class of prehomogeneous spaces can be constructed inside the tangent spaces to the rational homogeneous varieties: these are called parabolic homogeneous spaces. Prehomogeneous spaces are also related to interesting birational transformations. They appear naturally in the geometry of Severi and Scorza varieties.

Let G denote an affine algebraic group and V a G -module.

DEFINITION 3.1.1. The G -module V is a prehomogeneous vector space if it contains an open G -orbit $\mathcal{O}$. The isotropy subgroup G_x of a point $x \in \mathcal{O}$ is independent of x up to conjugacy. It is called the *generic isotropy group*.

Note that V remains prehomogeneous for any finite index subgroup of G . In particular we will suppose in the sequel that G is connected.

EXAMPLE 3.1.2. The group $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on the space $M_{m,n}(\mathbb{C})$ of matrices of size $m \times n$, and the set of matrices of maximal rank is an open orbit, so $M_{m,n}(\mathbb{C})$ is a prehomogeneous vector space. What is the generic isotropy group? More generally, take any G -module V , of dimension n . Then $V \otimes \mathbb{C}^m$ is a prehomogeneous vector space for $G \times GL_m(\mathbb{C})$ if $m \geq n$.

DEFINITION 3.1.3. A non zero rational function f on V is a *relative invariant* if there exists a character $\chi \in X(G)$ such that

$$f(gx) = \chi(g)f(x) \quad \forall g \in G, x \in V.$$

The hypersurface ($f = 0$) in V is then G -invariant, and since G is connected each of its irreducible component is invariant. The converse also holds.

Proposition 3.1.4. Let $f_1, \dots, f_m$ be equations of the irreducible components of $V/\mathcal{O}$. They are algebraically independent relative invariants of V , called the *fundamental invariants*, and every relative invariant is, up to scalar, of the form $f_1^{p_1} \cdots f_m^{p_m}$ for some integers $p_1, \dots, p_m \in \mathbb{Z}$.

PROOF. Since the hypersurface ($f_i = 0$) is G -invariant, for every $g \in G$, $f_i \circ g$ is again an equation of it, hence a multiple of f_i by some scalar $\chi_i(g)$, and χ_i is clearly a character. These characters are all different, because $\chi_i = \chi_j$ would imply that f_i/f_j is an invariant rational function, in particular constant on $\mathcal{O}$, hence on V . But by Dedekind's lemma, relative invariants with different characters are always linearly independent. Indeed, suppose with have a relation $a_1 f_1 + \cdots + a_m f_m = 0$, where the character of f_i is χ_i , and $\chi_1, \dots, \chi_m$ are pairwise distinct. Applying this relation to gx , with $g \in G$ and $x \in V$, we get the relations

$$a_1 \chi_1(g) f_1 + \cdots + a_m \chi_m(g) f_m = 0, \quad \forall g \in G.$$

Since pairwise distinct characters are linearly independent, this implies that $a_1 = \cdots = a_m = 0$. Moreover the same argument implies that any polynomial relation between $f_1, \dots, f_m$ must be trivial, that is, $f_1, \dots, f_m$ are algebraically independent.

Conversely, decompose any relative invariant into irreducible factors. Because G is connected, each of these factors is itself a relative invariant, and defines an irreducible component of $V - \mathcal{O}$. So it must be one of the f_i 's. $\square$

Proposition 3.1.5. *Suppose that the prehomogeneous vector space V is an irreducible G -module. Then V has at most one fundamental invariant.*

PROOF. Suppose we have two fundamental invariants f, g , of degrees d, e respectively. Then f^e/g^d is an invariant rational function, it must be constant: indeed, since V is irreducible, the center of G acts by homotheties on V , and the semisimple part has no non trivial character. But then V cannot be prehomogeneous. $\square$

EXAMPLE 3.1.6. The general linear group GL_n acts on $\mathbb{C}^n$ with only two orbits, the origin and its complement. In particular there is no relative invariant. This remains true for the action of Sp_n acts on $\mathbb{C}^n$ (n even). On the contrary $\mathbb{C}^n$ is not a prehomogeneous vector space for the action of SO_n , but it is one for $\mathbb{C}^* \times SO_n$, the invariant quadratic form q being a relative invariant.

EXAMPLE 3.1.7. The action of GL_n on the space of symmetric or skew-symmetric two-forms is again prehomogeneous: there are finitely many orbits classified by the rank (which is always even in the skew-symmetric case). The determinant provides a relative invariant, non trivial except in the skew-symmetric case and in odd dimensions. In this case there is no relative invariant, since all the orbits have codimension bigger than one (except the open one of course). In even dimensions the determinant is not a fundamental invariant, being the square of the Pfaffian, which is the fundamental invariant.

EXAMPLE 3.1.8 (Pencils of skew-symmetric forms). The strong difference between skew-symmetric forms in even and odd dimensions is of course also apparent when we consider the space $V = \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^n$, with the action of $G = GL_2 \times SL_n$. A tensor in V which is not decomposable defines a pencil of skew-symmetric forms, and studying such pencils is essentially equivalent to study SL_2 -orbits in V .

Since $\dim V = n(n-1) < \dim G = n^2 + 3$ we could expect V to be a prehomogeneous vector space. This cannot be the case if $n = 2m$ is even with $m > 3$. Indeed, consider an element of V as a map from $\mathbb{C}^2$ to $\wedge^2 \mathbb{C}^n$. Taking the Pfaffian we get a polynomial function of degree m on $\mathbb{C}^2$. When this polynomial is non zero it defines a m -tuple of points on $\mathbb{P}^1$. Since PSL_2 does not act generically transitively on such m -tuples of points, the action cannot be prehomogeneous.

Of course the argument does not apply when n is odd since the Pfaffian does not exist in this case. One can check that $V = \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^{2m+1}$ is in fact prehomogeneous under the action of $G = GL_2 \times SL_{2m+1}$. Indeed, a direct computation shows that the stabilizer of $v_1 \otimes \omega_1 + v_2 \otimes \omega_2$ has the correct dimension if

$$\begin{aligned}\omega_1 &= e_1 \wedge e_2 + \cdots + e_{2m-1} \wedge e_{2m}, \\ \omega_2 &= e_2 \wedge e_3 + \cdots + e_{2m} \wedge e_{2m+1}.\end{aligned}$$

Note that the G -orbit of $v_1 \otimes \omega_1 + v_2 \otimes \omega_2$ is open in V if and only if the PSL_n -orbit of $\langle \omega_1, \omega_2 \rangle$ is open in the Grassmannian $G(2, V)$ parametrizing pencils of skew-symmetric forms (this is an important remark that we will use again when we will define castling transforms). Therefore the generic pencil is equivalent $\langle \omega_1, \omega_2 \rangle$.

3.2. Regular invariants

Let f be a relative invariant on the prehomogeneous vector space V . The rational map

$$\begin{aligned}\phi_f : V &\dashrightarrow V^* \\ x &\mapsto df(x)/f(x)\end{aligned}$$

is G -equivariant (with respect to the dual action of G on V^*). Moreover it is well-defined on the open orbit $\mathcal{O}$. Its degree is -1 , and ϕ_f should be considered as a kind of inverse mapping (see the examples below).

The differential of ϕ_f at a point $x \in \mathcal{O}$ is a linear map from V to V^* , whose determinant we denote by $H(f)(x)$. In a given basis, this is just the usual determinant of the matrix of second derivatives of $\log(f)$ – its Hessian.

DEFINITION 3.2.1. A prehomogeneous vector space V is *regular* if it admits a relative invariant f whose Hessian $H(f)$ is not identically zero.

Proposition 3.2.2. *The prehomogeneous vector space V is regular if and only if there is a relative invariant f such that the rational map $\phi_f : V \dashrightarrow V^*$ is dominant. In particular, V^* is also prehomogeneous for the dual action of G .*

EXAMPLE 3.2.3. Consider the action on $V = \mathbb{C}^2$, of the stabilizer $G \subset GL_2(\mathbb{C})$ of some non zero vector. Check that this is a prehomogeneous vector space whose dual space is not prehomogeneous.

In the cases we will be interested in, the group G will be reductive. Most irreducible prehomogeneous vector space of reductive groups are well-behaved, in the sense that they have a unique non trivial relative invariant, and this relative invariant is regular. One can show that the existence of a regular fundamental invariant is equivalent to the condition that the complement of the open orbit is a hypersurface. This is in turn equivalent to the condition that the open orbit is affine, which by a theorem of Matsushita can be characterized by the property that the generic isotropy group is reductive (see [20], Theorem 2.28).

In practice this is an important criterion, since we often check that a vector space is a prehomogeneous vector space by computing the isotropy algebra of an explicit point. When we have done this computation, it is in general easy to decide whether this algebra is reductive or not. If it is reductive, we know that we have to go in search of a relative invariant. Then we face the problem that we don't know a priori what should be the degree of such an invariant. We will discuss this question in the last section, in connection with projective duality.

What we can quickly decide is the number of fundamental invariants that we need to find. Indeed, let v belong to the open orbit $\mathcal{O}$ in V , and let G_v denote its stabilizer (the generic stabilizer). If f is a non trivial invariant with character χ , the relation $f(gv) = \chi(g)f(v)$ implies that $\chi(G_v) = 1$. Conversely, if χ is a character of G with this property, we can define a regular function on $\mathcal{O}$ by letting $f(gv) = \chi(g)$ for any $g \in G$. This is a rational function on V , and its character χ is therefore the difference of the characters of two regular invariants.

Let us summarize this discussion.

Proposition 3.2.4. *Let V be a prehomogeneous vector space of a reductive group G , let v belong to the open orbit. Then V is regular if and only if the generic stabilizer G_v is reductive. If this is the case, the number of fundamental invariants is the rank of the group*

$$X(G, V) = \text{Ker}(X(G) \rightarrow X(G_v)).$$

EXAMPLE 3.2.5. Consider $V = \wedge^3 \mathbb{C}^6$, acted on by $G = GL_6$. A straightforward computation shows that the point $v = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$ has a stabilizer G_v whose connected component is isomorphic to $SL_3 \times SL_3$. In particular the G -orbit

of v is open. Moreover, since G_v is semi-simple and $X(G) = \mathbb{Z}$, the prehomogeneous vector space V is regular and there is only one fundamental invariant, hence the complement of the open orbit must be an irreducible hypersurface.

EXAMPLE 3.2.6. Let $G = \mathbb{C}^* \times SO_3 \times Sp_{2n}$ act on $V = \mathbb{C}^3 \otimes \mathbb{C}^{2n}$. We claim that this is a prehomogeneous space for any $n \geq 2$. A relative invariant can be constructed as follows. Let q be the quadratic form preserved by SO_3 and ω the symplectic form preserved by Sp_{2n} . We consider V as the space of maps v from $(\mathbb{C}^3)^*$ to $\mathbb{C}^{2n}$. Such a map will send q to an element of $S^2\mathbb{C}^{2n}$, hence q^2 to an element of $Q(v) \in S^2(S^2\mathbb{C}^{2n})$. But there is an Sp_{2n} -invariant linear form Ω on the latter space, defined by

$$\Omega(ab, cd) = \omega(a, c)\omega(b, d) + \omega(a, d)\omega(b, c).$$

We deduce a degree four relative invariant of G by letting $f(v) = \Omega(Q(v))$. In coordinates, suppose given an orthonormal basis (e_1, e_2, e_3) of $\mathbb{C}^3$ and let $v = e_1 \otimes v_1 + e_2 \otimes v_2 + e_3 \otimes v_3$. Then

$$\frac{1}{2}f(v) = \omega(v_1, v_2)^2 + \omega(v_2, v_3)^2 + \omega(v_3, v_1)^2.$$

This relative invariant is *not* regular (see [20], Example 2.30 page 70).

There is a nice interplay between a prehomogeneous vector space and its dual, that we will describe in some detail. The main motivation is a nice relation with Cremona transformations – birational transformations of projective spaces.

Over the complex numbers, a reductive group G is the complexification of a compact Lie group K (in fact, this is the original definition). For example $G = GL_n(\mathbb{C})$ is the complexification of the unitary group $K = U_n$.

Now, our complex representation V of G can always be endowed with a K -invariant Hermitian form; otherwise said, we may suppose it is contained in the unitary group. But then the dual representation V^* , restricted to K , is equivalent to the complex conjugate representation $\bar{V}$ since $\bar{g} = (g^{-1})^t$ when $g \in K$.

Suppose that V has a relative invariant f of degree $d \geq 2$. By letting

$$f^*(x) = \overline{f(\bar{x})}, \quad x \in V^*,$$

we obtain a relative invariant of K on $V^* \simeq \bar{V}$, of the same degree as f . Since G is the complexification of K , this will automatically be a relative invariant of G . Moreover, f^* is clearly regular as soon as f is.

3.3. Cremona transformations

A very nice property of regular invariants is that it provides interesting examples of Cremona transformations.

Theorem 3.3.1. *Suppose that G is reductive and that the prehomogeneous vector space V is regular, with a unique fundamental invariant f . Then $df : \mathbb{P}V \dashrightarrow \mathbb{P}V^*$ is a birational map, with inverse $df^* : \mathbb{P}V^* \dashrightarrow \mathbb{P}V$.*

PROOF. The main observation is that starting from the relative invariants f and f^* , both of degree d , we can construct a new relative invariant $F = f^*(df)$. Since f^* is regular F cannot be zero, and since its degree is $d(d-1)$ there exists a non zero constant c such that $F = 2cf^{d-1}$. The result will be obtained by differentiating this identity.

For $x \in \mathcal{O}$ consider $\varphi_f(x) = df(x)/f(x) \in V^*$. Its differential $d\varphi_f(x)$ is a linear map from V to V^* , and by the regularity hypothesis this map is invertible. For $\delta \in V^*$, we claim that

$$(3.3.1) \quad \langle x, d\varphi_f(x)(\delta) \rangle = -2\langle \delta, \varphi_f(x) \rangle.$$

This follows readily from the Euler identity, φ_f being homogeneous of degree -1 .

Now the relation $F = 2cf^{d-1}$ can be rewritten as $f^*(\varphi_f(x)) = 2cf(x)^{-1}$. Differentiating, we obtain

$$\langle df^*(\varphi_f(x)), d\varphi_f(x)(\delta) \rangle = -2cf(x)^{-2} \langle \delta, df(x) \rangle.$$

Taking 3.3.1 into account and letting $\theta = d\varphi_f(x)(\delta)$, this can be rewritten as

$$\langle df^*(\varphi_f(x)), \theta \rangle = cf(x)^{-1} \langle x, \theta \rangle.$$

But since $d\varphi_f(x)$ is invertible this is true for any θ , and we deduce that $df^*(\varphi_f(x)) = cf(x)^{-1}x$, or equivalently

$$df^*(df(x)) = cf(x)^{d-2}x.$$

This completes the proof. $\square$

EXAMPLE 3.3.2. Consider $G = (\mathbb{C}^*)^n$ acting diagonally on $V = \mathbb{C}^n$. This is a prehomogeneous space, the open orbit being the set of vectors with only non zero coordinates. The relative invariant $f(x) = x_1 \cdots x_n$ is regular, and the associated Cremona transformation is

$$\varphi_f(x) = (x_1^{-1}, \dots, x_n^{-1}).$$

EXAMPLE 3.3.3. Consider the action of $G = GL_n$ on $V = \text{Sym}^2 \mathbb{C}^n$, with its regular fundamental invariant given by the determinant. Since the differential of the determinant is the comatrix, which is proportional to the inverse, the associated Cremona transformation of $\mathbb{P}(\text{Sym}^2 \mathbb{C}^n)$ is just the inverse map. The same conclusion holds for the action of $G = GL_n \times GL_n$ on $V = \mathbb{C}^n \otimes \mathbb{C}^n$ and the action of $G = GL_{2n}$ on $V = \wedge^2 \mathbb{C}^{2n}$.

EXAMPLE 3.3.4. Consider $G = GL_2$ acting on $V = \text{Sym}^3(\mathbb{C}^2)^*$, the space of cubic polynomials. A direct computation shows that the stabilizer of the cubic $x^3 + y^3$ is finite, so that its orbit is open and V is prehomogeneous. More is true: the stabilizer being reductive this prehomogeneous space must be regular. We can construct a relative invariant f as follows: starting from a cubic C , its Hessian $He(C)$ is a quadratic polynomial, and we can let $f(C)$ equal its discriminant. Explicitely, for $C = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3$ we get

$$f(C) = 18a_1a_2a_3a_4 + a_2^2a_3^2 - 27a_1^2a_4^2 - 4a_1a_3^3 - 4a_2^3a_4.$$

The theorem implies that the rational map

$$(a_1, a_2, a_3, a_4) \mapsto \begin{pmatrix} 9a_2a_3a_4 - 27a_1a_4^2 - 2a_1a_3^3, 9a_1a_3a_4 + a_2a_3^2 - 6a_2^2a_4, \\ 9a_1a_2a_4 + a_2^2a_3 - 6a_1a_3^2, 9a_1a_2a_3 - 27a_1^2a_4 - 4a_2^3 \end{pmatrix}$$

is an involutive Cremona transformation of $\mathbb{P}^3$.

Using the same ideas as above, one can prove the following statement:

Theorem 3.3.5. *Let V be a prehomogeneous vector space of a reductive group G , with a relative polynomial invariant f , and let f^* be the corresponding relative invariant of V^* . Then there exists a polynomial b in one variable, of the same degree as f , such that*

$$f^*(\partial)f^s = b(s)f^{s-1} \quad \forall s \in \mathbb{Z}.$$

The polynomial b is called the *b-function* of the prehomogeneous vector space, and such polynomials have been studied extensively. One of the simplest examples is the case where $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on $V = M_{n,n}(\mathbb{C})$. We know this is a prehomogeneous vector space with a regular relative polynomial invariant given by the determinant. The corresponding b -function is $b(s) = s(s+1) \cdots (s+n-1)$, and the identity

$$\det\left(\frac{\partial}{\partial x_{ij}}\right) \det(x)^s = s(s+1) \cdots (s+n-1) \det(x)^{s-1}$$

was discovered by Cayley.

3.4. Castling transforms

In the sequel we only deal with reductive groups. An important idea in the classification of prehomogeneous vector spaces is to define equivalence classes. First, two prehomogeneous vector spaces V and V' on which the groups G and G' respectively act through the representations ρ and ρ' , are *strongly equivalent* if there is an isomorphism of V with V' that identifies the actions of $\rho(G)$ and $\rho'(G')$. This means that we only care with the effective action of $\rho(G)$ of V , not really of G .

A more subtle idea is to use the following result.

Proposition 3.4.1. *Let V be a G -module of dimension n , and p, q be integers such that $p+q=n$. Then $V \otimes \mathbb{C}^p$ is a prehomogeneous vector space for $G \times GL_p(\mathbb{C})$, if and only if $V^* \otimes \mathbb{C}^q$ is one for $G \times GL_q(\mathbb{C})$.*

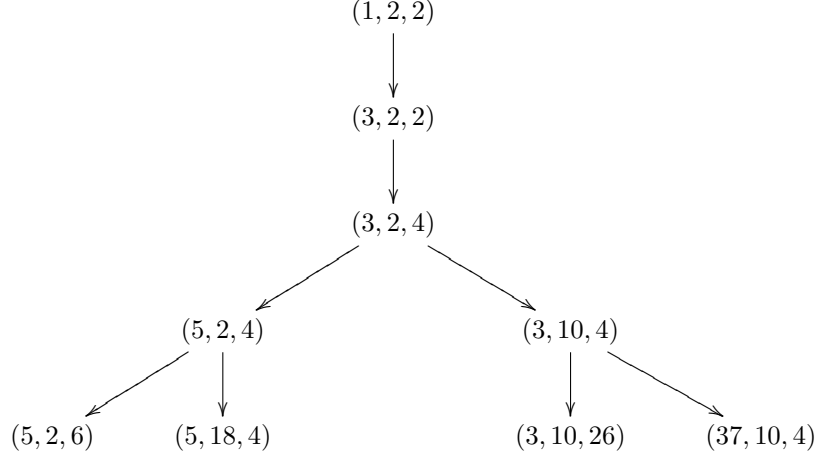
PROOF. Suppose that $V \otimes \mathbb{C}^p \simeq V \oplus \cdots \oplus V$ is a prehomogeneous vector space for $G \times GL_p(\mathbb{C})$. The open subset Ω of independent p -tuples $(v_1, \dots, v_p)$ is $G \times GL_p(\mathbb{C})$ -stable, hence contains the open orbit. Moreover, we have a map from Ω to the Grassmannian $G(p, V)$, sending $(v_1, \dots, v_p)$ to the p -dimensional subspace P of V that they span. Note that the fiber of P is the space of all its basis, on which $GL_p(\mathbb{C})$ acts transitively. Therefore, $G \times GL_p(\mathbb{C})$ has an open orbit in $V \otimes \mathbb{C}^p$, if and only if G has an open orbit in the Grassmannian $G(p, V)$.

Now we use that fact that by duality, $G(p, V) \simeq G(q, V^*)$, in such a way that the action on the former is identified with the dual action on the latter. So G has an open orbit on $G(p, V)$ if and only if it has an open orbit on $G(q, V^*)$, and by the same argument as above, this is equivalent to the fact that $V^* \otimes \mathbb{C}^q$ is a prehomogeneous vector space for $G \times GL_q(\mathbb{C})$. $\square$

REMARK 3.4.2. One can easily check that the generic isotropy groups are the same in $V \otimes \mathbb{C}^p$ and $V^* \otimes \mathbb{C}^q$. In particular, if one of these spaces is regular, the other one is also regular.

REMARK 3.4.3. If V is an n -dimensional prehomogeneous vector space for G , it is also prehomogeneous for the action of $G \times GL_1(\mathbb{C}) = G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts by homotheties. We conclude that $V^* \otimes \mathbb{C}^{n-1}$ is a prehomogeneous vector space for $G \times GL_{n-1}(\mathbb{C})$. Applying this procedure again and again, we can construct infinite series of prehomogeneous vector spaces from a given one.

EXAMPLE 3.4.4. Starting from the obvious fact that the action of $GL_1(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ on $\mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is prehomogeneous. Let us denote $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ by (a, b, c) for simplicity. By castling transforms we obtain the following sequence of prehomogeneous vector spaces :



DEFINITION 3.4.5. Two irreducible prehomogeneous vector spaces U and U' are related by a *castling transform* if there is an n -dimensional irreducible G -module V , and integers p, q with $p + q = n$, such that U is strongly isomorphic to $V \otimes \mathbb{C}^p$ and U' to $V^* \otimes \mathbb{C}^q$.

Two irreducible prehomogeneous vector spaces U and U' are *equivalent* if they can be related by a finite sequence of castling transforms.

Proposition 3.4.6. *Every equivalence class of prehomogeneous vector spaces contains a unique element of minimal dimension, up to duality.*

PROOF. We can prove a stronger statement, which is that if two prehomogeneous vector spaces are related by a sequence of castling transforms, then the dimensions of the sequence of prehomogeneous vector spaces produced by these transforms must be monotonous. Suppose the contrary. Then we can find a prehomogeneous vector space V for a group G with two different castling transforms of smaller dimensions. These transforms need to correspond to two distinct factors in G , so we may suppose that $G = H \times GL_a \times GL_b$ and $V = U \otimes \mathbb{C}^a \otimes \mathbb{C}^b$. Denote by u the dimension of the H -module U . The two castling transforms are then $U \otimes \mathbb{C}^a \otimes \mathbb{C}^{ua-b}$ and $U \otimes \mathbb{C}^{ub-a} \otimes \mathbb{C}^b$. If they are both of dimension smaller than V , then $ua < 2b$ and $ub < 2a$, therefore $u < 2$, a contradiction. $\square$

3.5. The case of tensor products

We have seen that using castling transforms we can produce infinite sequences of tensor products $\mathbb{C}^{k_1} \otimes \dots \otimes \mathbb{C}^{k_m}$ for which the action of $GL_{k_1}(\mathbb{C}) \times \dots \times GL_{k_m}(\mathbb{C})$ is prehomogeneous. We would like to know which of these tensor products have finitely many orbits.

Lemma 3.5.1. *Let V be a G -module, with G reductive. If $V \otimes \mathbb{C}^n$ has finitely $G \times GL_n(\mathbb{C})$ orbits and $n \leq \dim V$, then $V \otimes \mathbb{C}^m$ has finitely $G \times GL_m(\mathbb{C})$ orbits for all $m \leq n$.*

PROOF. Again we consider $V \otimes \mathbb{C}^n$ as the space of n -tuples $(v_1, \dots, v_n)$, and we have a stratification by the dimension k of the span of these n -vectors. Each strata can be mapped to a Grassmannian $G(k, V)$, and using the same argument as in the proof of Proposition 3.4.1, there is a bijective correspondence between the $G \times GL_n(\mathbb{C})$ -orbits in $V \otimes \mathbb{C}^n$ and the G -orbits in the disjoint union of the $G(k, V)$ for $0 \leq k \leq n$. The claim follows immediately. $\square$

Proposition 3.5.2. *Suppose $k_1 \geq \dots \geq k_m > 1$ and $m \geq 3$.*

If $GL_{k_1}(\mathbb{C}) \times \dots \times GL_{k_m}(\mathbb{C})$ has finitely many orbits in the tensor product $\mathbb{C}^{k_1} \otimes \dots \otimes \mathbb{C}^{k_m}$, then $m = 3$, $k_3 = 2$ and $k_2 \leq 3$.

PROOF. By the previous lemma, it is enough to prove that $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ have infinitely many orbits. Since the center of each copy of $GL_2(\mathbb{C})$ acts by homotheties, the orbits of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ are the same as those of $\mathbb{C}^* \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. But this group has dimension 13, so cannot have an open orbit in the 16-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The second case can be treated in the same way. We leave the last case as an exercise: one has to prove that the generic isotropy group is five-dimensional. $\square$

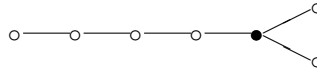
It turns out that there is no other obstruction.

Theorem 3.5.3. *The group $GL_{k_1}(\mathbb{C}) \times \dots \times GL_{k_m}(\mathbb{C})$ has finitely many orbits in the tensor product $\mathbb{C}^{k_1} \otimes \dots \otimes \mathbb{C}^{k_m}$, if and only if $m \leq 2$, or $m = 3$ and $(k_1, k_2, k_3) = (n, 2, 2)$ or $(n, 3, 2)$.*

Parfenov has studied the orbits and their incidence relations. A remarkable fact is that the orbit structure stabilizes when n is large enough, as indicated by the following table, which includes the degree of the fundamental relative invariant, when there is one.

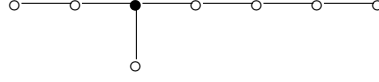
(k_1, k_2, k_3)	# orbits	deg f
$(2, 2, 2)$	7	4
$(3, 2, 2)$	9	6
$(4, 2, 2)$	10	4
$(n \geq 5, 2, 2)$	10	$\emptyset$
$(3, 3, 2)$	18	12
$(4, 3, 2)$	24	12
$(5, 3, 2)$	26	$\emptyset$
$(6, 3, 2)$	27	6
$(n \geq 7, 3, 2)$	27	$\emptyset$

Most of these prehomogeneous vector spaces are parabolic (see the next lecture). For example, the isotropic Grassmannian $G_Q(n-2, 2n)$ is encoded in the weighted Dynkin diagram



The corresponding space with finitely many orbits is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{n-2}$.

The cases of type $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$ come from the exceptional groups up to $n = 5$, which corresponds to the E_8 -Grassmannian with weighted Dynkin diagram



REMARK 3.5.4. Note that having finitely many orbits is a much stronger condition than being prehomogeneous vector space. In particular it is not preserved by casting transforms. Nevertheless, it follows from the classification theorems of Sato and Kimura that any prehomogeneous tensor product is casting equivalent to a prehomogeneous tensor product with finitely many orbits.

3.6. Relations with projective duality

There are several recipes that allow to compute the degree of the fundamental invariant of a simple prehomogeneous space V . Kimura gives a trace formula which boils down to a linear algebra computation (see [20], Proposition 2.19 page 34).

More geometrically, consider a G -orbit closure Z in the projective space $\mathbb{P}V$. The projective dual variety Z^* inside $\mathbb{P}V^*$ is G -stable, hence a G -orbit closure since $\mathbb{P}V^*$ has only finitely many orbits. Since projective duality is involutive, we deduce that:

Proposition 3.6.1. *Projective duality defines a bijection between G -orbits in $\mathbb{P}V$ and G -orbits in $\mathbb{P}V^*$.*

Beware that this bijection is in general rather badly behaved. For example it is not compatible with the closure ordering on orbits.

In general, the projective dual of a variety $Z \subset \mathbb{P}V$ is expected to be a hypersurface, unless Z has some special properties (for example Z will be uniruled). Of course orbit closures are quite special and the proposition shows that their duals are almost never hypersurfaces. Nevertheless, one could hope that a minimal G -orbit (closure) Z_{min} in $\mathbb{P}V$ has a codimension one dual variety. Its equation would then be a fundamental (semi)invariant of V^* .

For a smooth projective variety $Z \subset \mathbb{P}V$ of dimension n , there exist quite explicit formulas that allow to compute the degree of the dual variety, and decide whether it is a hypersurface or not. Denote by h the hyperplane class on Z and let

$$\delta_Z = \sum_{i \geq 0} (i+1) \int_Z c_{n-i}(\Omega_Z) h^i.$$

If $\delta_Z = 0$, then the dual variety Z^* is not a hypersurface. If $\delta_Z \neq 0$, then Z^* is a hypersurface of degree δ_Z . In particular in the case where Z is the minimal G -orbit in a simple prehomogeneous space, then δ_Z is the degree of the fundamental invariant.

Even if $\delta_Z = 0$, one can determine the dimension and the degree of the dual variety as follows. Consider the polynomial

$$P_Z(q) = \sum_{i \geq 0} q^{i+1} \int_Z c_{n-i}(\Omega_Z) h^i.$$

Proposition 3.6.2 (Katz). *Let c be the minimal integer such that $P_Z^{(c)}(1) \neq 0$. Then Z^* has codimension c and degree $P_Z^{(c)}(1)/c!$.*

EXAMPLE 3.6.3. Consider $G = GL_n \times GL_n$ acting on $V = \mathfrak{gl}_n \simeq M_n(\mathbb{C})$ by multiplication on the left and on the right. The orbits are defined by the rank, so that the closure ordering is complete. Moreover projective duality exchanges the rank with the corank. In particular the minimal orbit is $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, and its dual is the determinant hypersurface.

The same remark applies to symmetric or skew-symmetric matrices. In the latter case, note that the prehomogeneous space $V = A_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ is regular only when $n = 2m$ is even. The minimal orbit in $\mathbb{P}V$ is the Grassmannian $G(2, 2m)$, whose dual variety is the Pfaffian hypersurface, of degree m .

For n odd, V has no semi-invariant and $\mathbb{P}V$ contains no invariant hypersurface. The complement of the open orbit is the dual variety of $G(2, 2m+1)$. This is a codimension three subvariety of $\mathbb{P}V^*$, of degree $2m+1$.

EXAMPLE 3.6.4. Consider $G = GL_5$ or SL_5 acting on $V = \wedge^2 \mathbb{C}^5$. There are only three orbits, parametrizing tensors of rank 0, 2 or 4. In particular the Grassmannian $G(2, 5)$ must be *self-dual*.

The tensor product $\mathbb{C}^k \otimes V$ remains prehomogeneous under the action of $GL_k \times GL_5$ for $k \leq 4$. The minimal orbit Z in $\mathbb{P}(\mathbb{C}^k \otimes V)$ is $\mathbb{P}^{k-1} \times G(2, 5)$. Its codimension c and degree δ are as follows:

k	c	δ
1	3	5
2	2	15
3	1	15
4	1	40

EXAMPLE 3.6.5. We have shown how to produce infinite sequences of prehomogeneous spaces of type $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$ acted on by $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_m}(\mathbb{C})$. We can even impose that these tensor products are regular with a unique fundamental invariant. How to define this invariant is not clear in general. As we suggested above, we can try to use the fact that the projectivization $\mathbb{P}(\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m})$ contains $\mathbb{P}(\mathbb{C}^{k_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{k_m})$, whose dual variety is expected to be a hypersurface. This is known to hold when the m -tuple $(k_1, \dots, k_m)$ is *balanced* in the sense that $2k_i \leq k_1 + \cdots + k_m$ for each $1 \leq i \leq m$. When this is true the degree of the dual hypersurface, hence of the fundamental invariant, is computed in [14]. Unfortunately the condition that the m -tuple $(k_1, \dots, k_m)$ is balanced is in general not preserved by a casting transformation.

CHAPTER 4

Prehomogeneous spaces of parabolic type

There is a close connection between parabolic subgroups of simple Lie groups and a large family of prehomogeneous vector spaces. These will be defined by gradings of semisimple Lie algebras.

4.1. Classification of $\mathbb{Z}$ -gradings

DEFINITION 4.1.1. A $\mathbb{Z}$ -grading of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In particular, $\mathfrak{g}_0$ is a subalgebra and each component $\mathfrak{g}_i$ is a $\mathfrak{g}_0$ -module.

Our main example is the following. Take $\mathfrak{g}$ semisimple, $\mathfrak{t}$ a Cartan subalgebra and Δ a set of simple roots. Choose a subset $I \subset \Delta$, and let $H_I \in \mathfrak{t}$ be defined by the conditions that

$$\alpha(H_I) = 0 \quad \text{when } \alpha \in I, \quad \alpha(H_I) = 1 \quad \text{when } \alpha \notin I.$$

This element H_I induces a $\mathbb{Z}$ -grading of $\mathfrak{g}$ defined by letting

$$\mathfrak{g}_i = \{X \in \mathfrak{g}, [H_I, X] = iX\}.$$

Note that a root space $\mathfrak{g}_\beta$, with $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$, is contained in $\mathfrak{g}_i$ for $i = \sum_{\alpha \notin I} n_\alpha$. In particular, the subalgebra

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in \langle I \rangle} \mathfrak{g}_\alpha,$$

where $\langle I \rangle \subset \Delta$ is the set of roots that are linear combinations of the simple roots in I . The center of $\mathfrak{g}_0$ is $\mathfrak{t}_I = \{H \in \mathfrak{t}, \alpha(H) = 0 \forall \alpha \in I\}$, and the derived algebra

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \bigoplus_{\alpha \in I} \mathbb{C}H_\alpha \oplus \bigoplus_{\alpha \in \langle I \rangle} \mathfrak{g}_\alpha$$

is semisimple. Its Dynkin diagram can be deduced from that of $\mathfrak{g}$ just by erasing the vertices corresponding to the simple roots not belonging to I .

More generally, we can associate a $\mathbb{Z}$ -grading to any element $H \in \mathfrak{t}$ such that $\alpha_i(H) \in \mathbb{Z}$ for any simple root α_i .

Proposition 4.1.2. *Up to conjugation, any $\mathbb{Z}$ -grading of the semisimple Lie algebra $\mathfrak{g}$ is defined by $H \in \mathfrak{t}$, and one can suppose that $\alpha_i(H) \in \mathbb{Z}_+$.*

Remark. Let us return to the previous situation. Then $\mathfrak{p}_I = \bigoplus_{i \geq 0} \mathfrak{g}_i$ is the Lie algebra of the parabolic subgroup P_I of G . Moreover, we get a filtration of $\mathfrak{g}$ by $\mathfrak{p}_I$ -modules if we let $\mathfrak{g}_{\geq j} = \bigoplus_{i \geq j} \mathfrak{g}_i$. The homogeneous bundle defined by the adjoint action of P_I on $\mathfrak{g}_{\geq 1}$ is isomorphic with the cotangent bundle of G/P_I .

4.2. Parabolic prehomogeneous spaces

Let us suppose for simplicity that I is the complement of a single simple root, which means that P_I is a maximal parabolic subgroup of G , and G/P_I a generalized Grassmannian.

Proposition 4.2.1. *In this case, each component $\mathfrak{g}_i$ of the grading, for $i \neq 0$, is an irreducible $\mathfrak{g}_0$ -module.*

PROOF. The main observation is that $\mathfrak{g}_i$ is a sum of root spaces in $\mathfrak{g}$ with the same coefficient on the simple root which is not in I . This means that, considered as a module over the semisimple part of $\mathfrak{g}_0$, each $\mathfrak{g}_i$ has only weights of multiplicity one, which are all congruent with respect to its root lattice.

This is enough to ensure the irreducibility. Indeed, suppose that we have two highest weights μ and ν , congruent with respect to the root lattice. Looking at the signs of the coefficients of $\mu - \nu$ expressed in terms of simple roots, we see that there exist two disjoint sets J, K of simple roots, and two positive linear combinations μ_J, ν_K of simple roots in J, K , respectively, such that $\mu - \mu_J = \nu - \nu_K$. Call this weight λ . If $i \notin J$,

$$\langle \lambda, \alpha_i^\vee \rangle = \langle \mu - \mu_J, \alpha_i^\vee \rangle \geq 0$$

since $\langle \mu, \alpha_i^\vee \rangle \geq 0$ and $\langle \alpha_j, \alpha_i^\vee \rangle \leq 0$ for any $j \neq i$. Similarly $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ when $i \notin K$. But then this holds for any i , hence λ is dominant. Being smaller than μ it must be a weight of V_μ , and also of V_ν for the same reason. But then it would have multiplicity at least two in $\mathfrak{g}_i$, a contradiction. $\square$

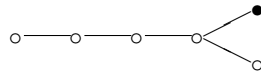
The case of $\mathfrak{g}_1$ is particularly simple, since we know in this case that α_i is the only lowest weight, if i does not belong to I . This allows to determine pictorially the representation $\mathfrak{g}_1$ of the semisimple part of $\mathfrak{g}_0$.

Remark. A particularly nice case is when $\mathfrak{g}_{\geq 2} = 0$. Then $\mathfrak{g}$ has a three-step grading

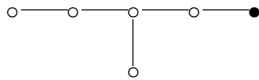
$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

hence a particularly simple structure. The generalized Grassmannian G/P_I is then called a *cominuscule* homogeneous space.

This happens exactly when the highest root has coefficient one on some of the simple roots. Therefore one can easily list all the cominuscule homogeneous spaces by looking at the highest roots of the simple Lie algebras. For example, from the Dynkin diagrams of types D_6 and E_6 we get the following gradings:



$$\mathfrak{so}_{12} = \wedge^2(\mathbb{C}^6)^* \oplus \mathfrak{gl}_6 \oplus \wedge^2(\mathbb{C}^6)$$



$$\mathfrak{e}_6 = \Delta_- \oplus (\mathbb{C} \oplus \mathfrak{so}_{10}) \oplus \Delta_+$$

Let G_0 denote the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. The adjoint action of G on $\mathfrak{g}$, when restricted to G_0 , stabilizes the subspaces $\mathfrak{g}_i$.

Theorem 4.2.2 (Vinberg). *The action of G_0 on $\mathfrak{g}_1$ has a finite number of orbits. In particular, it is prehomogeneous.*

DEFINITION 4.2.3. A prehomogeneous vector space obtained by this procedure will be called a parabolic prehomogeneous vector space.

PROOF. The idea is to deduce this statement from the finiteness of nilpotent orbits in $\mathfrak{g}$. Indeed, $\mathfrak{g}_1$ is contained in the nilpotent cone of $\mathfrak{g}$, so it is enough to prove that any nilpotent G -orbit $\mathcal{O}$ in $\mathfrak{g}$ intersects $\mathfrak{g}_1$ along the union of finitely many

G_0 -orbits. We will prove a more precise statement: each irreducible component of $\mathcal{O} \cap \mathfrak{g}_1$ is a G_0 -orbit.

This follows from an infinitesimal computation. Let x belong to $\mathcal{O} \cap \mathfrak{g}_1$. The tangent space at x to this (schematic) intersection is $T_x \mathcal{O} \cap \mathfrak{g}_1 = \mathfrak{g}.x \cap \mathfrak{g}_1$. The tangent space to the G_0 -orbit of x is $\mathfrak{g}_0.x$. We just need to check that they are equal. Obviously $\mathfrak{g}.x \cap \mathfrak{g}_1 \supset \mathfrak{g}_0.x$.

Conversely, let $X \in \mathfrak{g}$, and decompose it as $\sum_k X_k$ according to the grading. For $Y = [X, x]$ to belong to $\mathfrak{g}_1$, we need that $[X_k, x] = 0$ for $k \neq 0$. But then $Y = [X_0, x]$ belongs to $\mathfrak{g}_0.x$. $\square$

4.3. Classification of orbits

The previous statement does not really provides us with a convenient tool for classifying the G_0 -orbits in $\mathfrak{g}_1$. Indeed, the intersection of a nilpotent G -orbits with $\mathfrak{g}_1$ can be very non transverse; it will often be empty, and when it is not, the number of its connected component seems difficult to control a priori. That is why other classification schemes have been developped, more in the spirit of Bala-Carter theory. The following definition is due to Vinberg.

DEFINITION 4.3.1. A graded reductive subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is:

- *regular* if it is normalized by a Cartan subalgebra of $\mathfrak{g}_0$; then $\mathfrak{s}$ is the direct sum of a subspace of this Cartan subalgebra with some of the root vectors – in particular regular subalgebras can be classified combinatorially;
- *complete* if it is not contained in a bigger graded reductive subalgebra $\mathfrak{s}'$ of $\mathfrak{g}$ of the same rank;
- *locally flat* if $\mathfrak{s}_0$ and $\mathfrak{s}_1$ have the same dimension; an important example is obtained by taking any semisimple Lie algebra $\mathfrak{s}$, and letting $\mathfrak{s}_0$ be a Cartan subalgebra and $\mathfrak{s}_1$ be the sum of the simple root spaces.

Theorem 4.3.2. *There is a bijection between the G_0 -orbits in $\mathfrak{g}_1$ and the G_0 -conjugacy classes of graded semi-simple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ that are regular, complete and locally flat.*

Let us explain how this bijection is defined.

Starting from an element e in $\mathfrak{g}_1$, we start by completing it into a $\mathfrak{sl}_2$ -triple (e, h, f) , with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. Consider in G_0 the normalizer $N_0(e)$ of the line generated by e . For any element u of its Lie algebra, there is a scalar $\varphi(u)$ such that

$$[u, e] = \varphi(u)e.$$

In particular $\varphi(h) = 2$. Let H be a maximal torus in $N_0(e)$, whose Lie algebra $\mathfrak{h}$ contains h . Define, for each integer k ,

$$\mathfrak{g}_k(\mathfrak{h}) = \{x \in \mathfrak{g}_k, \quad [u, x] = k\varphi(u)x \quad \forall x \in \mathfrak{h}\}.$$

The direct sum $\mathfrak{g}(\mathfrak{h})$ of these spaces is a graded subalgebra of $\mathfrak{g}$. By construction $\mathfrak{g}_0(\mathfrak{h}) = \mathfrak{z}_0(\mathfrak{h}) \supset \mathfrak{h}$ and $e \in \mathfrak{g}_1(\mathfrak{h})$. One can show that $\mathfrak{g}(\mathfrak{h})$ is always reductive. Let $\mathfrak{s}$ be its semi-simple part, called the *support* of e . Vinberg proves that $\mathfrak{s}$ is regular, complete and locally flat. Moreover the S_0 -orbit of e is dense in $\mathfrak{s}_1$ (this is where the flatness condition comes from).

The latter property shows how to define the reverse bijection. Starting from a graded semi-simple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, which is regular, complete and locally flat,

we simply choose for e a generic element in $\mathfrak{s}_1$ – more precisely, an element of the open S_0 -orbit.

So we have replaced the problem of classifying G_0 -orbits in $\mathfrak{g}_1$ by the problem of classifying G_0 -conjugacy classes of graded semi-simple regular, complete and locally flat subalgebras $\mathfrak{s}$ of $\mathfrak{g}$. As already noted, the regularity property implies that this problem can be treated as a problem about root systems, hence purely combinatorial.

Concretely, one can start with $\mathfrak{s}$, an abstract graded locally flat semi-simple algebra, with a Cartan subalgebra and a root space decomposition. Then we try to embed it in $\mathfrak{g}$ as a regular subalgebra. This embedding will be determined by the images of the simple root vectors of $\mathfrak{s}$. Up to conjugation it will be determined by associating to each of these simple roots α_i a root β_i in the root system of $\mathfrak{g}$. These roots need not be simple, but they have to share the same pattern as the simple roots of $\mathfrak{s}$ – this will be enough to guarantee that there is a compatible embedding of $\mathfrak{s}$ inside $\mathfrak{g}$.

One important ingredient that we still miss is the classification of regular, complete and locally flat graded semi-simple algebras. This classification was obtained by Vinberg. The most important class is given by the *principal* gradings, which are obtained as follows. Consider any semi-simple Lie algebra $\mathfrak{s}$, choose a Cartan subalgebra $\mathfrak{t}$ and a set of simple roots. Then define a grading of $\mathfrak{s}$ by letting $\mathfrak{s}_0 = \mathfrak{t}$ and $\mathfrak{s}_1$ equals the sum of the simple root spaces. Obviously this grading is regular and locally flat, it is also complete.

Some non principal gradings can be obtained by giving degree zero to some of the simple roots, and degree one to the remaining ones. It is easy to see that the local flatness condition implies that the vertex of the Dynkin diagram corresponding to the degree zero root is a triple vertex, which is quite restrictive (we will call these non principal gradings elementary). This indicates that the non principal gradings are relatively scarce and can be classified.

EXAMPLE 4.3.3. Start with $\mathfrak{g} = \mathfrak{e}_7$ with the grading defined by the node at the end of the shortest arm of the Dynkin diagram. This is a five-step grading, namely

$$\mathfrak{e}_7 = \mathbb{C}^7 \oplus \wedge^3(\mathbb{C}^7)^* \oplus \mathfrak{gl}_7 \oplus \wedge^3(\mathbb{C}^7) \oplus (\mathbb{C}^7)^*.$$

The GL_7 -orbits in $\wedge^3(\mathbb{C}^7)$ have been first determined by Schouten in 1931. There are exactly ten orbits including zero. As was already known to E. Cartan, the generic stabilizer is (up to a finite group) a form of the exceptional group G_2 . Moreover a normal form (different from the one below) of the generic three-form encodes the multiplication table of the Cayley octonion algebra.

The correspondence between orbits and graded subalgebras allows us to exhibit a representative ω of each orbit. This goes as follows, where we denote by $\mathcal{O}^d$ the unique orbit of dimension d .

(Note that this contains the classification of GL_6 -orbits inside $\wedge^3(\mathbb{C}^6)$: there are exactly four orbits of dimensions 0, 10, 15, 19, 20. We will come back to this example later on.)

$\mathcal{O}$	$\mathfrak{s}$	ω
$\mathcal{O}^{35}$	$3A_1 \times A_2$	$e_{123} + e_{145} + e_{167} + e_{246} + e_{357}$
$\mathcal{O}^{34}$	$2A_1 \times A_2$	$e_{123} + e_{145} + e_{246} + e_{357}$
$\mathcal{O}^{31}$	$A_1 \times A_2$	$e_{123} + e_{246} + e_{357}$
$\mathcal{O}^{28}$	$4A_1$	$e_{123} + e_{145} + e_{167} + e_{246}$
$\mathcal{O}^{26}$	A_2	$e_{123} + e_{456}$
$\mathcal{O}^{25}$	$3A_1$	$e_{123} + e_{145} + e_{246}$
$\mathcal{O}^{24}$	$3A_1$	$e_{123} + e_{145} + e_{167}$
$\mathcal{O}^{20}$	$2A_1$	$e_{123} + e_{145}$
$\mathcal{O}^{13}$	A_1	e_{123}

EXAMPLE 4.3.4. Let us consider the grading of $\mathfrak{e}_6$ defined by the central vertex of the Dynkin diagram of type E_6 . Then we need to determine what can be the restriction Q the Killing form of $\mathfrak{e}_6$ to the weights of $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, which we represent as triples (ijk) with $1 \leq i \leq 2$ and $1 \leq j, k \leq 3$. We do not need to make an explicit computation, which could be tedious: we just need to observe that the result must be invariant by the Weyl group of $\mathfrak{g}_0$, which is the product of the permutation groups of the indices i, j and k respectively. Moreover, since $\mathfrak{e}_6$ is simply-laced, we know that we can normalize Q in such a way that $Q(ijk, ijk) = 2$ and $Q(ijk, lmn) \in \{-1, 0, 1\}$ if ijk and lmn are distinct. The only possibility is that

$$Q(ijk, lmn) = \delta_{il} + \delta_{jm} + \delta_{kn} - 1.$$

Then we must first analyze the embeddings of the semisimple Lie algebras with their principal gradings. concretely we take a Dynkin diagram and try to associate to each node a basis vector of $\mathfrak{g}_1$, in such a way that the scalar products match. We get the following lists of possibilities, up to isomorphism:

A_1	(111)
$2A_1$	(111)(122)
$2A_1$	(111)(212)
$2A_1$	(111)(221)
A_2	(111)(222)
$3A_1$	(111)(122)(133)
$3A_1$	(111)(212)(221)
$A_2 \times A_1$	(111)(222)(123)
$A_2 \times A_1$	(111)(222)(132)
A_3	(111)(222)(133)
$A_2 \times 2A_1$	(111)(222)(123)(132)
$A_2 \times 2A_1$	(111)(222)(123)(213)
$A_2 \times 2A_1$	(111)(222)(132)(231)
$2A_2$	(111)(222)(123)(231)
$A_3 \times A_1$	(111)(222)(133)(213)
$2A_2 \times A_1$	(111)(222)(123)(231)(132)

Then there remains to deal with the non principal gradings. One can check that this gives only one additional case, coming from an elementary non principal grading of type D_4 , that we denote $D_4(\alpha_2)$.

$$D_4(\alpha_2) \quad (111)(122)(133) - \alpha_1$$

where α_1 is the positive root of the $\mathfrak{sl}_2$ factor.

Finally we get 18 orbits in $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ (including the origin) and we can provide explicit representatives.

It is rather remarkable that the classification of orbits in these two spaces of tensors, and a few more, are in fact controlled by the exceptional Lie algebras!

4.4. The closure ordering

The Zariski closure on a G_0 -orbit $\mathcal{O}$ in $\mathfrak{g}_1$ is a union of G_0 -orbits. We define the *closure ordering* on the set of orbits by letting

$$\mathcal{O} \geq \mathcal{O}' \quad \text{when} \quad \overline{\mathcal{O}} \supset \mathcal{O}'.$$

It is an interesting problem to determine the closure ordering. Unfortunately it is not known how to interpret it in terms of supports.

EXAMPLE 4.4.1. In the space of matrices $M_{m,n}$ with its usual action of $GL_m \times GL_n$, the orbits $\mathcal{O}_r$ are classified by the rank r . Since the rank is semi-continuous, $\mathcal{O}_r \geq \mathcal{O}_s$ if and only if $r \geq s$.

In simple cases it is possible to define semi-continuous invariants: typically when we deal with tensors we can consider the ranks of auxiliary maps that can be constructed. This is sometimes sufficient to determine completely the closure ordering.

EXAMPLE 4.4.2. Let us come back to the case $\wedge^3(\mathbb{C}^7)$, whose GL_7 -orbits we have just described. One can define two auxiliary invariants, the rank and the 2-rank, that we are going to describe.

Each element $\omega \in \wedge^3(\mathbb{C}^7)$ defines a map

$$a_\omega : \wedge^2(\mathbb{C}^7) \rightarrow \wedge^5(\mathbb{C}^7) \simeq \wedge^2(\mathbb{C}^7)^*,$$

and the rank of ω is the rank of this map. The definition of the 2-rank is more subtle. Observe that there exists a GL_7 -equivariant map $S^3(\wedge^3(\mathbb{C}^7)) \rightarrow S^2\mathbb{C}^7$ defined up to constant. Indeed, let us choose a generator Ω of $\det(\mathbb{C}^7)$. For $u \in (\mathbb{C}^7)^*$ denote by $\omega_u \in \wedge^3(\mathbb{C}^7)$ the contraction of ω by u . We define a quadratic form q_ω on $(\mathbb{C}^7)^*$ by the formula

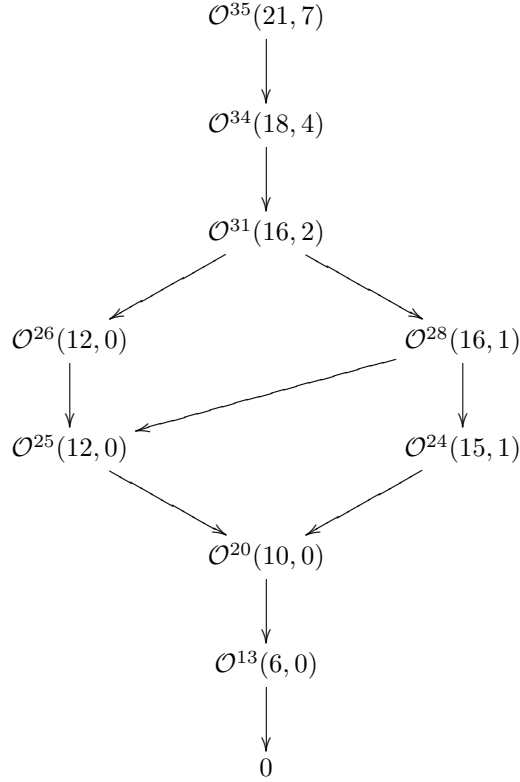
$$\omega \wedge \omega_u \wedge \omega_u = q_\omega(u) \Omega.$$

The 2-rank is the rank of this quadratic form. Note that q_ω is non-degenerate if and only if ω belongs to the open orbit, and this yields the classical embedding $G_2 \subset SO_7$.

Since the rank and the 2-rank cannot increase by specialization, they impose strong restrictions of the possibilities for an orbit to belong to the closure of an open orbit. Note also that the determinant of a_ω defines a semi-invariant of degree 21. However, since the rank drops by three on the codimension one orbit, this determinant must be a cube, and indeed the fundamental invariant has degree

seven. It can also be computed as the degree of the dual hypersurface to the Grassmannian $G(3, 7)$.

With the help of these two invariants, and a little extra work, one shows that the closure ordering on orbits is represented by the following diagram. Here we denoted the orbits by $\mathcal{O}^d(r, \rho)$ rather than simply $\mathcal{O}^d$, with r the 2-rank and ρ the rank.



In more complicated cases, invariants like the rank or the two-rank of the previous example are not necessary available, or they do not allow to determine the closure ordering completely.

4.5. Desingularizations of orbit closures

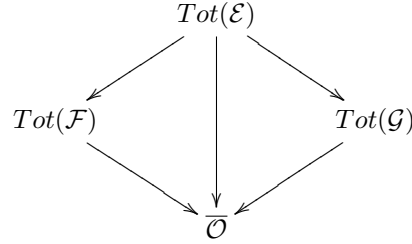
Let $\mathcal{O}$ be the G_0 -orbit of some element $e \in \mathfrak{g}_1$, and let us complete it into a $\mathfrak{sl}_2$ -triple (e, h, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. The semi-simple element h acts on $\mathfrak{g}_0$ and $\mathfrak{g}_1$ with integer eigenvalues. The direct sum of its non-negative eigenspaces in $\mathfrak{g}_0$ is a parabolic subalgebra $\mathfrak{p}$; the direct sum in $\mathfrak{g}_1$ of its eigenspaces associated to eigenvalues at least equal to two is a $\mathfrak{p}$ -submodule $\mathfrak{u}$ of $\mathfrak{g}_1$. Let P denote the parabolic subgroup of G_0 with Lie algebra $\mathfrak{p}$.

Proposition 4.5.1. *The natural map $\pi : G_0 \times^P \mathfrak{u} \rightarrow \mathfrak{g}_1$ defines a G_0 -equivariant resolution of singularities of $\overline{\mathcal{O}}$.*

This can be used to determine the closure ordering.

Corollary 4.5.2. *Let $\mathcal{O}'$ be another G_0 -orbit in $\mathfrak{g}_1$. Then $\mathcal{O} \geq \mathcal{O}'$ if and only if $\mathcal{O}'$ meets $\mathfrak{u}$.*

EXAMPLE 4.5.3. Let us come back to the case of $\wedge^3 \mathbb{C}^6$ and its five orbit closures. We have seen that the unique GL_6 -orbit $\mathcal{O}$ of codimension five is represented by the point $\omega = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5$. Let U_1 and U_5 be the tautological vector bundles of rank one and five over the flag variety $\mathbb{F}_{1,5}$, and let $\mathcal{E}$ denote the vector bundle $U_1 \wedge (\wedge^2 U_5)$ over $\mathbb{F}_{1,5}$. The total space of $\mathcal{E}$ projects to $\wedge^3 \mathbb{C}^6$ and gives the canonical desingularization of $\overline{\mathcal{O}}$. But there are two other natural desingularizations, obtained as follows. Denote by V_1 the tautological line bundle on $\mathbb{P}(\mathbb{C}^6)$, and by V_5 the tautological hyperplane bundle on the dual $\mathbb{P}(\mathbb{C}^6)^*$. Let $\mathcal{F}$ be the vector bundle $V_1 \wedge (\wedge^2 \mathbb{C}^6)$ over $\mathbb{P}(\mathbb{C}^6)$, and let $\mathcal{G}$ be the vector bundle $\wedge^3 V_5$ over $\mathbb{P}(\mathbb{C}^6)^*$. Then both the total spaces of $\mathcal{F}$ and $\mathcal{G}$ are desingularizations of $\overline{\mathcal{O}}$, and there is a diagram



and the birational transformation between $Tot(\mathcal{F})$ and $Tot(\mathcal{G})$ is a flop. In particular the canonical desingularization of $\overline{\mathcal{O}}$ is not minimal. One can check that the two other desingularizations are crepant, while the canonical one is not.

It is a rather general phenomenon that a G_0 -orbit closure in $\mathfrak{g}_1$ has many different equivariant resolutions of singularities by total spaces of vector bundles (and even more alterations). Understanding which are the best behaved is an open problem.

4.6. Classification of irreducible prehomogeneous vector spaces

The classification of irreducible prehomogeneous vector spaces of reductive complex algebraic groups has been obtained by Sato and Kimura [29]. Recall that we used casting transforms to define an equivalence relation on prehomogeneous vector spaces, and that each equivalence class contains a unique *reduced* prehomogeneous vector space, that is, of minimal dimension.

Theorem 4.6.1. *Every irreducible prehomogeneous vector space V of a reductive complex algebraic group G is casting equivalent to either:*

- (1) *a trivial prehomogeneous vector space, that is, some $U \otimes \mathbb{C}^n$ acted on by $H \times SL_n$, where U is any irreducible representation of the reductive group H , of dimension smaller than n ;*
- (2) *a parabolic prehomogeneous vector space;*
- (3) *the restriction of a parabolic prehomogeneous vector space to some subgroup,*
- (4) *$GL_{2m+1} \times SL_2$ or $SL_{2m+1} \times SL_2$ acting on $\wedge^2 \mathbb{C}^{2m+1} \otimes \mathbb{C}^2$.*

This follows only a posteriori from the explicit classification. Let us be more precise.

We can list all the parabolic prehomogeneous vector spaces by listing the Dynkin diagrams and their vertices. Each case provides us with a reductive group

G_0 with center $\mathbb{C}^*$, with a prehomogeneous action on $V = \mathfrak{g}_1$. Most of them are regular with the following exceptions:

G_0	V	G	P
GL_{2m+1}	$\wedge^2 \mathbb{C}^{2m+1}$	D_{2m+1}	α_{2m+1}
$GL_{2m+1} \times SL_2$	$\wedge^2 \mathbb{C}^{2m+1} \otimes \mathbb{C}^2$	E_{2m+2}	α_2
$GL_{2m+1} \times Sp_{2n}$	$\mathbb{C}^{2m+1} \otimes \mathbb{C}^{2n}$	C_{n+2m+1}	α_{2m+2}
$\mathbb{C}^* \times Spin_{10}$	Δ_+	E_6	α_1

Here in the column denoted by G we give the type of the simple Lie group whose Lie algebra admits a grading yielding the prehomogeneous vector space (G_0, V) . In the column P we give the simple root that defines the grading. Those prehomogeneous vector spaces have no non trivial relative invariant. Note that the case coming from E_{2m+2} is parabolic only for $m \leq 3$, but yields prehomogeneous vector spaces for arbitrary m !

The fact that there is no relative invariant can be shown to imply that the action remains prehomogeneous when restricted to a subgroup which is a complement to the center of G_0 – in each case a copy of $\mathbb{C}^*$.

There remains only seven “accidental” cases obtained by restriction from G_0 to a subgroup H . The list is the following:

G_0	V	G	P	H
$GL_2 \times SO_8$	$\mathbb{C}^2 \otimes \mathbb{C}^8$	D_6	α_2	$GL_2 \times Spin_7$
$GL_3 \times SO_8$	$\mathbb{C}^3 \otimes \mathbb{C}^8$	D_7	α_3	$GL_3 \times Spin_7$
$\mathbb{C}^* \times SO_{16}$	$\mathbb{C}^{16}$	D_9	α_1	$\mathbb{C}^* \times Spin_9$
$\mathbb{C}^* \times Spin_{12}$	$\mathbb{C}^{32}$	E_7	α_1	$\mathbb{C}^* \times Spin_{11}$
$\mathbb{C}^* \times SO_7$	$\mathbb{C}^7$	B_4	α_1	$\mathbb{C}^* \times G_2$
$GL_2 \times SO_7$	$\mathbb{C}^2 \otimes \mathbb{C}^7$	B_5	α_2	$GL_2 \times G_2$

These cases have the same flavor: they come from special low-dimensional embeddings of simple Lie groups in slightly bigger groups, namely $Spin_7 \subset SO_8$ and $Spin_9 \subset SO_{16}$ (these embeddings are defined by the spin representations of $Spin_7$ and $Spin_9$, the natural embedding $Spin_{11} \subset Spin_{12}$, and the embedding $G_2 \subset SO_7$, which will be the starting point of the next lecture.

About the proof. A substantial part of the proof of the classification theorem by Sato and Kimura consists in listing all the irreducible G -modules V such that $\dim G \geq \dim V$, which is an obvious condition for V to be prehomogeneous. The Weyl dimension formula is used to express the dimension of an irreducible module in a convenient way, in terms of its highest weight. Of course we can eliminate the trivial prehomogeneous vector spaces and the non reduced ones. Then it is not difficult to show that G cannot have more than three simple factors. After some work Sato and Kimura obtain an explicit finite list. Then they consider each case one by one and manage to decide, essentially by explicit computations, which is prehomogeneous and which is not.

4.7. $\mathbb{Z}_m$ -graded Lie algebras

There is a nice variant of the theory that is, in some way, closer to the case of nilpotent orbits we started with. We have explained how to classify $\mathbb{Z}$ -gradings of semisimple Lie algebras. It is natural to ask about gradings over more general groups or monoids. We will focus on $\mathbb{Z}_m$ -gradings.

The first observation is that defining a $\mathbb{Z}_m$ -grading on a Lie algebra $\mathfrak{g}$ is equivalent to giving a group homomorphism f from $\mathbb{Z}_m$ to $\text{Aut}(\mathfrak{g})$: the graded parts of $\mathfrak{g}$ being the eigenspaces of $\theta = f(1)$. If $\mathfrak{g}$ is semisimple, the algebraic group $\text{Aut}(\mathfrak{g})$ is a finite extension of the group of inner automorphisms, which is the adjoint group G (the quotient is the group of outer automorphisms, it is isomorphic with the automorphism group of the Dynkin diagram). The $\mathbb{Z}_m$ -grading of $\mathfrak{g}$ is called *inner* if $\theta = \text{Ad}(h)$ is an inner automorphism. Since $\theta^m = 1$, the element $h \in G$ is semisimple and its eigenvalues are m -th roots of unity. One can show that up to conjugation, such an h can be prescribed by the following recipe.

Let us suppose that $\mathfrak{g}$ is simple. Choose as usual a Cartan subalgebra and a set of simple roots Δ . Let ψ denote the highest root.

DEFINITION 4.7.1. The *affine Dynkin diagram* of $\mathfrak{g}$ is obtained by adding to the Dynkin diagram a vertex (representing $-\psi$), and connecting it to the vertex representing the simple root α_i by $\langle \psi, \alpha_i^\vee \rangle$ edges (oriented as usual from the long to the short root in the non simply laced case).

Let n_i be the coefficient of ψ on the simple root α_i , and let $n_0 = 1$. Then up to conjugation, the inner automorphism θ is uniquely defined by non-negative integers θ_0 and $\theta_i, i \in \Delta$, such that

$$n_0\theta_0 + \sum_{i \in \Delta} n_i\theta_i = m.$$

For example, from the affine Dynkin diagrams of types E_7 and E_8 we get the following gradings, over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively:

$$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ - \circ \\ | \\ \bullet \end{array} \quad \mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \wedge^4(\mathbb{C}^8)$$

$$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ - \circ - \circ \\ | \\ \bullet \end{array} \quad \mathfrak{e}_8 = \wedge^3(\mathbb{C}^9)^* \oplus \mathfrak{sl}_9 \oplus \wedge^3(\mathbb{C}^9)$$

An important difference with the case of $\mathbb{Z}$ -gradings is that a subspace $\mathfrak{g}_i, i \neq 0$, need not be contained in the nilpotent cone. Nevertheless, the same argument as in the $\mathbb{Z}$ -graded case shows that there are only finitely many G_0 -orbits of nilpotent elements in $\mathfrak{g}_i$. Moreover they can again be classified by their support, and their Zariski closures admit equivariant desingularizations by total spaces of vector bundles.

For example nilpotent orbits in $\wedge^4(\mathbb{C}^8)$ and $\wedge^3(\mathbb{C}^9)$ can be classified.

CHAPTER 5

The magic square

5.1. The exceptional group G_2 and the octonions

5.1.1. Stabilizers of three-forms. The classical groups $O(V)$, $Sp(V)$, $SL(V)$ are all defined to be groups preserving some generic tensor. In addition to these, there is just one more class of simple group that can be defined as the group preserving a generic tensor of some type on a vector space. The reason there are so few is that the tensor spaces $S_\pi V$ almost always have dimension greater than that of $SL(V)$, and then the subgroup of $SL(V)$ preserving a generic element will be zero dimensional.

REMARK 5.1.1. The group preserving a generic tensor is not always simple, for example the *odd symplectic group* Sp_{2n+1} preserving a generic $\omega \in \Lambda^2 \mathbb{C}^{2n+1}$ is not even reductive. However it does have interesting properties.

Other than S^2V and Λ^2V , the only examples of Schur powers $S_\pi V$ of smaller dimension than $SL(V)$ are Λ^3V (and their duals) for $n = 6, 7$ and 8 . We expect the generic stabilizers to be groups respectively of dimensions $16, 14$ and 8 . Note that Λ^3V , for $n = 6, 7, 8$, is a parabolic prehomogeneous vector space coming from the shortest arm of the Dynkin diagram of E_n .

$n = 6$. We have already obtained a normal form for the generic element ω in $\Lambda^3 \mathbb{C}^6$: in a well-chosen basis we can write it as

$$\omega = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6.$$

The subgroup of GL_6 fixing ω must preserve the pair of vector spaces $\langle e_1, e_2, e_3 \rangle$, $\langle e_4, e_5, e_6 \rangle$. We thus get for generic stabilizer an extension of $SL_3 \times SL_3$ by $\mathbb{Z}_2$.

$n = 8$. We begin with a general construction. Let $\mathfrak{g}$ be a complex Lie algebra with Killing form K . We define the *Cartan 3-form* $\phi \in \Lambda^3 \mathfrak{g}^*$ by the formula

$$\phi(X, Y, Z) = K(X, [Y, Z]), \quad X, Y, Z \in \mathfrak{g}.$$

This is a $\mathfrak{g}$ -invariant 3-form. Moreover, if $\mathfrak{g}$ is semisimple the infinitesimal stabilizer of the Cartan 3-form is exactly $\mathfrak{g}$.

Suppose that $\mathfrak{g} = \mathfrak{sl}_3$, which has dimension eight. Up to a finite group the stabilizer of the Cartan 3-form ϕ is SL_3 , in particular the dimension of its GL_8 -orbit is $\dim GL_8 - \dim SL_3 = 64 - 8 = 56$. So ϕ must belong to the open orbit in $\Lambda^3(\mathfrak{sl}_3)^*$ and the generic stabilizer is a finite extension of SL_3 .

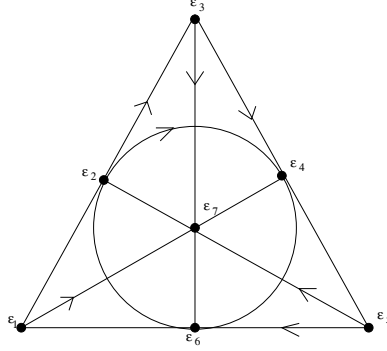
5.1.2. The exceptional group G_2 . In the case $n = 7$, first note that a form $\phi \in \Lambda^3 V^*$ and a volume form $\Omega \in \Lambda^7 V^*$ determine a quadratic form as follows. For $v \in V$ denote by $\phi_v \in \Lambda^2 V^*$ the contraction of ϕ by v . Then we define $q_\phi \in S^2 V^*$ by the identity

$$\phi_v \wedge \phi_w \wedge \phi = q_\phi(v, w)\Omega, \quad v, w \in V.$$

For a generic ϕ , the symmetric bilinear form q_ϕ will be non degenerate. In particular the stabilizer of ϕ is a subgroup of the orthogonal group $O(q_\phi) \simeq O_7$.

A little more calculation (see [15], Theorem 6.80) shows that one obtains a simple Lie group of dimension 14 . In the Cartan-Killing classification there is only one such group, G_2 .

These two cases are a bit disappointing, since the generic stabilizers are well-known groups. The case $n = 7$ turns out to be more interesting.



5.1.3. The octonions. The group G_2 is commonly defined in terms of *octonions*. The Cayley algebra of octonions is an eight-dimensional non-commutative and non-associative algebra of the real numbers, that we denote by $\mathbb{O}$. It admits a basis $e_0, e_1, \dots, e_7$ in which the multiplication is particularly simple: $e_0 = 1$ is the unit, and for $i \neq j \geq 1$,

$$e_i^2 = -1, \quad e_i e_j = \pm e_k$$

where k is determined by the diagram above¹.

This must be understood as follows: in the diagram any two vectors e_i and e_j are joined by a unique line (including the central circle), and there is a unique third vector on the line, which is e_k ; the sign is $+1$ if the arrow goes from e_i to e_j , and -1 otherwise; in particular $e_i e_j = -e_j e_i$.

Note that each of the four-dimensional subspaces generated by (e_0, e_i, e_j, e_k) is a copy of the more familiar Hamilton algebra of quaternions $\mathbb{H}$, which is non commutative but is associative.

One of the most remarkable properties of the octonion algebra is that it turns out to be a *normed* algebra, in the sense that it admits a norm such that the norm of a product is the product of the norms. The norm $|x|$ of $x = x_0 e_0 + \dots + x_7 e_7$ is defined by $|x|^2 = x_0^2 + \dots + x_7^2 = q(x)$. We also let $x \mapsto \bar{x}$ be the orthogonal symmetry with respect to the unit vector. Then $x + \bar{x} = 2x_0 e_0$ and x_0 is called the real part of x . Octonions with zero real part are called imaginary, and we denote by $\text{Im}(\mathbb{O})$ the space of imaginary octonions, spanned by $e_1, \dots, e_7$.

Proposition 5.1.2. *The Cayley algebra has the following properties.*

- (1) $|xy| = |x| \times |y| \quad \forall x, y \in \mathbb{O}$.
- (2) $\mathbb{O}$ is alternative, in the sense that the subalgebra generated by any two elements is associative.
- (3) As a consequence, the associator $\omega(x, y, z) = (xy)z - x(yz)$ is skew-symmetric.

It is a classical theorem that there exists only four normed algebras over the field of real numbers, forming the series $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$.

¹This diagram is usually called the Fano plane: it represents the projective plane over $\mathbb{F}_2$, with its seven points and seven lines.

Proposition 5.1.3. *The automorphism group of $\mathbb{O}$ is a compact connected Lie group of type G_2 .*

Note that every automorphism of $\mathbb{H}$ is inner, so that $\text{Aut}(\mathbb{H}) \simeq SO_3$. Also $\text{Aut}(\mathbb{C}) \simeq \mathbb{Z}_2$ (conjugation is the only non trivial automorphism) and $\text{Aut}(\mathbb{R}) = 1$.

Any automorphism of $\mathbb{O}$ fixes the unit element and preserves the norm. Therefore it preserves $\text{Im}(\mathbb{O})$, and it is naturally a subgroup of SO_7 . Moreover it preserves the three-form

$$\phi(x, y, z) = \text{Re}((xy)z - x(yz)).$$

Note that ϕ expressed in the canonical basis of $\text{Im}(\mathbb{O})$ has exactly seven terms, one for each line in the Cayley plane.

One can check that the stabilizer of this form is exactly $\text{Aut}(\mathbb{O})$. In particular ϕ is a generic three form in seven variables.

5.1.4. Triality. Given any algebra $\mathcal{A}$, its *automorphism group*

$$\text{Aut}(\mathcal{A}) := \{g \in GL(\mathcal{A}) \mid g(xy) = g(x)g(y) \ \forall x, y \in \mathcal{A}\}$$

is an affine algebraic group. The associated Lie algebra is the *derivation algebra*

$$\text{Der}(\mathcal{A}) := \{X \in \mathfrak{gl}(\mathcal{A}) \mid X.(xy) = (X.x)y + x(X.y) \ \forall x, y \in \mathcal{A}\}.$$

We may enlarge $\text{Aut}(\mathcal{A})$ to define the *triality group* of $\mathcal{A}$,

$$T(\mathcal{A}) := \{(g_1, g_2, g_3) \in GL(\mathcal{A})^{\times 3} \mid g_1(xy) = g_2(x)g_3(y) \ \forall x, y \in \mathcal{A}\}.$$

When $\mathcal{A}$ is equipped with a quadratic form, we require that $(g_1, g_2, g_3) \in SO(\mathcal{A})^{\times 3}$.

One also has the corresponding trialiality algebra

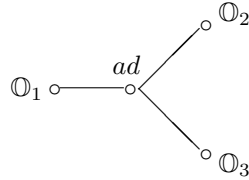
$$\mathfrak{t}(\mathcal{A}) := \{(X_1, X_2, X_3) \in \mathfrak{so}(\mathcal{A})^{\times 3} \mid X_1(xy) = (X_2(x))y + x(X_3(y)) \ \forall x, y \in \mathcal{A}\}.$$

This definition, due to B. Allison [1], generalizes the most important case of $\mathcal{A} = \mathbb{O}$, which is due to Elie Cartan.

There are three natural actions of $T(\mathcal{A})$ (or $\mathfrak{t}(\mathcal{A})$) on $\mathcal{A}$ corresponding to the three projections to $GL(\mathcal{A})$, and we denote these representations by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$.

Proposition 5.1.4 (Cartan's triality principle). *The triality group of the Cayley algebra is $T(\mathbb{O}) \simeq \text{Spin}_8$, each projection $T(\mathbb{O}) \rightarrow SO(\mathbb{O}_i)$, for $i = 1, 2, 3$, being a double covering.*

Moreover the representations $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ are non equivalent: they are the three fundamental representations of Spin_8 other than the adjoint representation, and they are permuted by the outer automorphisms of $T(\mathbb{O})$. This is encoded in the triple symmetry of the Dynkin diagram of type D_4 :



5.2. The magic square

Once we have constructed G_2 as the automorphism groups of the octonions, it is natural to ask whether we can construct the other exceptional groups in a similar fashion. In the 1950's, Freudenthal and Tits found a way to attach a Lie algebra to a pair of normed algebras. When one of the algebras is $\mathbb{O}$, one obtains the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$. The first construction that will be explained below is different, and more elementary than the original construction of Tits and Freudenthal, but it will yield in the end exactly the same Lie algebras.

5.2.1. The magic square from triality. Let $\mathbb{A}$ and $\mathbb{B}$ be two normed algebras. We will also allow the possibility that $\mathbb{B} = 0$. Consider the vector space

$$\mathfrak{g} = \mathfrak{g}(\mathbb{A}, \mathbb{B}) = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3).$$

We define a Lie algebra structure, that is, a Lie bracket on $\mathfrak{g}(\mathbb{A}, \mathbb{B})$, by the following conditions:

- $\mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B})$ is a Lie subalgebra, and the bracket with an element of $\mathbb{A}_i \otimes \mathbb{B}_i$ is given by the natural action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_i$ and $\mathfrak{t}(\mathbb{B})$ on $\mathbb{B}_i$;
- the bracket of two elements in $\mathbb{A}_i \otimes \mathbb{B}_i$ is given by the natural map

$$\wedge^2(\mathbb{A}_i \otimes \mathbb{B}_i) = \wedge^2 \mathbb{A}_i \otimes S^2 \mathbb{B}_i \oplus S^2 \mathbb{A}_i \otimes \wedge^2 \mathbb{B}_i \rightarrow \wedge^2 \mathbb{A}_i \oplus \wedge^2 \mathbb{B}_i \rightarrow \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}),$$

where the first arrow follows from the quadratic forms given on $\mathbb{A}_i$ and $\mathbb{B}_i$, and the second arrow is dual to the map $\mathfrak{t}(\mathbb{A}) \rightarrow \wedge^2 \mathbb{A}_i \subset \text{End}(\mathbb{A}_i)$ (and similarly for $\mathbb{B}$) prescribing the action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_i$ (which, by definition, preserves the quadratic form on $\mathbb{A}_i$). Here duality is taken with respect to a $\mathfrak{t}(\mathbb{A})$ -invariant quadratic form on $\mathfrak{t}(\mathbb{A})$, and the quadratic form on $\wedge^2 \mathbb{A}_i$ induced by that on $\mathbb{A}_i$;

- finally, the bracket of an element of $\mathbb{A}_i \otimes \mathbb{B}_i$ with one of $\mathbb{A}_j \otimes \mathbb{B}_j$, for $i \neq j$, is given by the following rules, with obvious notations:

$$\begin{aligned} [u_1 \otimes v_1, u_2 \otimes v_2] &= u_1 u_2 \otimes v_1 v_2 \in \mathbb{A}_3 \otimes \mathbb{B}_3, \\ [u_2 \otimes v_2, u_3 \otimes v_3] &= u_3 \bar{u}_2 \otimes v_3 \bar{v}_2 \in \mathbb{A}_1 \otimes \mathbb{B}_1, \\ [u_3 \otimes v_3, u_1 \otimes v_1] &= \bar{u}_1 u_3 \otimes \bar{v}_1 v_3 \in \mathbb{A}_2 \otimes \mathbb{B}_2. \end{aligned}$$

Theorem 5.2.1. *This defines a structure of semi-simple Lie algebra on $\mathfrak{g}(\mathbb{A}, \mathbb{B})$.*

Note that the construction works over the real or the complex numbers as well. Let us work over the complex numbers to simplify the discussion. Letting $\mathbb{A}, \mathbb{B}$ equal $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we obtain the following square of Lie algebras, call the *Tits-Freudenthal magic square*:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

There exists a rather strange formula for the dimensions of the $\mathfrak{g}(\mathbb{A}, \mathbb{B})$ in terms of $a = \dim \mathbb{A}$ and $b = \dim \mathbb{B}$, namely

$$\dim \mathfrak{g}(\mathbb{A}, \mathbb{B}) = 3 \frac{(ab + 4a + 4b - 4)(ab + 2a + 2b)}{(a + 4)(b + 4)}.$$

Formulas of that type were first obtained by Vogel and Deligne using ideas from knot theory. See [24] for more on this.

5.2.2. The magic square from Jordan algebras. The original version of the magic square was not obviously symmetric. Given a normed algebra $\mathbb{A}$, Freudenthal and Tits considered the space $\mathcal{J}_3(\mathbb{A})$ of 3×3 Hermitian matrices with coefficients in $\mathbb{A}$:

$$\mathcal{J}_3(\mathbb{A}) = \left\{ \begin{pmatrix} r & x & y \\ \bar{x} & s & z \\ \bar{y} & \bar{z} & t \end{pmatrix}, \quad r, s, t \in \mathbb{R}, \quad x, y, z \in \mathbb{A} \right\}.$$

There is a natural algebra structure on $\mathcal{J}_3(\mathbb{A})$: although the product of two Hermitian matrices is not in general Hermitian, the symmetrized product

$$X.Y = \frac{1}{2}(XY + YX), \quad X, Y \in \mathcal{J}_3(\mathbb{A}),$$

will be. This product is of course commutative, and although it is not associative it has some nice properties.

Proposition 5.2.2. *The algebra $\mathcal{J}_3(\mathbb{A})$ is a Jordan algebra: for any $X \in \mathcal{J}_3(\mathbb{A})$, the operators of multiplication by X and X^2 commute.*

Any Jordan algebra J is power associative, which means that the subalgebra generated by any element $X \in J$ is associative. In particular the powers X^m are well-defined. One defines the rank of J as the dimension of this subalgebra for X generic.

One can check that $\mathcal{J}_3(\mathbb{A})$ has rank three. Therefore there exists a polynomial identity

$$X^3 - f_1(X)X^2 + f_2(X)X - f_3(X)I = 0, \quad \forall X \in \mathcal{J}_3(\mathbb{A}),$$

where f_i is a polynomial of degree i . When $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ this is of course the usual formula. In particular $f_3(X)$ is the determinant, and we will use the notation $\text{Det}(X) = f_3(X)$ for any $X \in \mathcal{J}_3(\mathbb{A})$ and any $\mathbb{A}$.

The Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is called the exceptional Jordan algebra. Its first connection with the exceptional Lie groups is the following result of Chevalley.

Theorem 5.2.3. *The automorphism group of the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is a Lie group of type F_4 .*

This implies that the derivation algebra $\text{Der}(\mathcal{J}_3(\mathbb{O}))$ is simple of type $\mathfrak{f}_4$. The Tits-Freudenthal construction expands this identity in order to obtain the other exceptional Lie algebras. For a pair $\mathbb{A}, \mathbb{B}$ of normed algebras we let

$$\mathfrak{g}^{TF}(\mathbb{A}, \mathbb{B}) = \text{Der}(\mathbb{A}) \times \text{Der}(\mathcal{J}_3(\mathbb{B})) \oplus \text{Im}(\mathbb{A}) \otimes \mathcal{J}_3(\mathbb{B})_0,$$

where $\mathcal{J}_3(\mathbb{B})_0 \subset \mathcal{J}_3(\mathbb{B})$ is the hyperplane of traceless matrices. The resulting square of Lie algebras is again the Tits-Freudenthal magic square. In fact we can enlarge it a little bit by allowing $\mathbb{B}$ to be zero, in which case we take for $\mathcal{J}_3(\mathbb{B})$ the space of diagonal matrices – or we could even take $\mathcal{J}_3(\mathbb{B}) = 0$, which we denote formally $\mathbb{B} = \Delta$. With this notation we have

$$\mathfrak{g}^{TF}(\mathbb{A}, \Delta) = \text{Der}(\mathbb{A}), \quad \mathfrak{g}^{TF}(\mathbb{A}, 0) = \mathfrak{t}(\mathbb{A}).$$

We get (over the complex numbers) the following *magic rectangle*:

	Δ	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	0	0	$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	0	$\mathbb{C}^3$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{so}_3$	$\mathfrak{so}_3^3$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{g}_2$	$\mathfrak{so}_8$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

5.3. The geometry of the magic rectangle

In a long series of papers, Freudenthal showed that to each ordered pair of normed algebras $\mathbb{A}, \mathbb{B}$, one can associate a *special geometry*, with a group of natural transformations whose Lie algebra is precisely $\mathfrak{g}^{TF}(\mathbb{A}, \mathbb{B})$. The type of this geometry depends only on $\mathbb{A}$, for example for $\mathbb{A} = \mathbb{C}$ one obtains a plane projective geometry. This geometry is defined in terms of certain objects (points and lines for a plane projective geometry) which are parametrized by homogeneous varieties, and may have certain incidence relations (typically, a line can contain a point, or not). These homogeneous varieties have especially interesting properties.

We will take a different perspective, more in the spirit of our study of parabolic prehomogeneous vector spaces. Indeed we have noticed the relations between parabolic prehomogeneous vector spaces and tangent spaces of homogeneous spaces. Geometrically, the most interesting tangents lines are those that remain contained in the initial variety. So we introduce the following definition.

DEFINITION 5.3.1. Let $X = G/P \subset \mathbb{P}V$ be a projective homogeneous space, considered in its minimal equivariant embedding. The *reduction* of X is the variety $Y \subset \mathbb{P}T_x X$ of tangent directions to lines through x contained in X , where x is a given point of X .

Of course Y does not depend on x by homogeneity. By reduction we will construct series of interesting varieties. Our starting point will be the exceptional *adjoint varieties*.

5.3.1. Adjoint varieties. Let $\mathfrak{g}$ be a simple Lie algebra.

DEFINITION 5.3.2. The adjoint variety of $\mathfrak{g}$ is the projectivization of the minimal nilpotent orbit,

$$X_{ad}(\mathfrak{g}) = \mathbb{P}\mathcal{O}_{min} \subset \mathbb{P}\mathfrak{g}.$$

In other words, $X_{ad}(\mathfrak{g})$ is the only closed orbit inside $\mathbb{P}\mathfrak{g}$, for the action of the adjoint group. The adjoint varieties of the classical Lie algebras are the following.

$$\begin{array}{ll} SL_n & \mathbb{F}_{1,n-1} = \mathbb{P}(T\mathbb{P}^{n-1}), \\ SO_n & OG(2, n), \\ Sp_{2n} & v_2(\mathbb{P}^{2n-1}). \end{array}$$

Being the projectivization of conic symplectic varieties, the adjoint varieties have an induced structure, called a *contact structure*.

DEFINITION 5.3.3. A contact structure on a variety X is a hyperplane distribution with the following property. If the distribution is given by a hyperplane bundle H of the tangent bundle, and if L denotes the quotient line bundle, then the skew-symmetric bilinear map

$$\omega : \wedge^2 H \rightarrow L$$

induced by the Lie bracket is non degenerate.

Each nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is a symplectic cone, so its projectivization $\mathbb{P}\mathcal{O} \subset \mathbb{P}\mathfrak{g}$ is a contact variety (see [4] for more details). The adjoint variety is the only one which is closed.

Once we have fixed a Cartan subalgebra and a set of positive roots, we get a point of $X_{ad}(\mathfrak{g}) = \mathbb{P}\mathcal{O}_{min}$ by taking the root space of the highest root ψ .

Lemma 5.3.4. *For any root α one has $\alpha(H_\psi) \leq 2$, with equality if and only if $\alpha = \psi$.*

Another way to understand this statement is the following: if we consider the five-step grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

defined by H_ψ , then $\mathfrak{g}_2 = \mathfrak{g}_\psi$ is only one-dimensional. The parabolic algebra stabilizing the line $\mathfrak{g}_\psi$ is $\mathfrak{p} = \mathfrak{g}_{\geq 0}$. The tangent bundle of $X_{ad}(\mathfrak{g}) = G/P$ is the homogeneous bundle associated to the P -module $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Note that $\mathfrak{g}_{-1}$ is a P -submodule (but not $\mathfrak{g}_{-2}$). The exact sequence

$$0 \rightarrow \mathfrak{g}_{-1} \rightarrow \mathfrak{g}/\mathfrak{p} \rightarrow \mathfrak{g}_{-2} \rightarrow 0$$

of P -modules induces an exact sequence of vector bundles on $X_{ad}(\mathfrak{g})$

$$0 \rightarrow H \rightarrow TX_{ad}(\mathfrak{g}) \rightarrow L \rightarrow 0,$$

in particular a hyperplane distribution in X . Moreover the skew-symmetric bilinear map $\omega : \wedge^2 H \rightarrow L$ can be identified at our special point with the restriction of the Lie bracket $\wedge^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$. In particular it must be non-degenerate, since $\mathfrak{g}_{-1}$ is irreducible.

Projective contact manifolds are expected to be rather uncommon. The projectivization $\mathbb{P}(\Omega_Y)$ of the cotangent bundle of any smooth projective variety has a contact structure inherited from the natural symplectic structure on Ω_Y . But if we impose some restrictions, for example that the Picard number is one, only few examples are known. An important conjecture is the following.

Conjecture (Le Brun – Salamon) *Any Fano contact variety is the adjoint variety of a simple Lie algebra.*

So there would be three series of classical Fano contact varieties: the projective spaces of odd dimensions, the projectivized cotangent bundles the projective spaces, and the Grassmannians of lines contained in a smooth quadric. Plus five exceptional examples coming from the exceptional Lie algebras – their dimensions being 5, 15, 21, 33, 57.

5.3.2. Legendrian varieties. Now we apply the reduction procedure we have introduced above. We choose a point in the adjoint variety, and then we look at lines passing through that point and contained in $X_{ad}(\mathfrak{g}) = \mathbb{P}\mathcal{O}_{min}$. It is a general fact that the tangent directions to the lines must be contained inside the contact distribution, so we get a variety $Y \subset \mathbb{P}\mathfrak{g}_{-1}$. Recall that $\mathfrak{g}_{-1}$ is equipped with a ($\mathfrak{g}_{-2}$ -valued) symplectic form induced by the Lie bracket (in particular it must be even dimensional). A subspace of $\mathfrak{g}_{-1}$ is called *Lagrangian* if it is isotropic and of maximal dimension for this property (that is, half the dimension of $\mathfrak{g}_{-1}$).

Proposition 5.3.5. *The variety $Y \subset \mathbb{P}\mathfrak{g}_{-1}$ is Legendrian: each of its affine tangent spaces is a Lagrangian subspace of $\mathfrak{g}_{-1}$.*

Of course we know that $\mathbb{P}\mathfrak{g}_{-1}$ has only finitely many G_0 -orbits. Moreover Y is the only closed one. The orbit structure is in fact quite simple. Apart from Y and $\mathbb{P}\mathfrak{g}_{-1}$ itself there are only two orbit closures, namely the dual hypersurface Y^* of Y , and its singular locus $\sigma_+(Y)$, with the simplest possible closure ordering:

$$Y \subset \sigma_+(Y) \subset Y^* \subset \mathbb{P}\mathfrak{g}_{-1}.$$

An equation of the dual hypersurface of Y , which has degree four, can be defined in terms of the five-step grading of $\mathfrak{g}$: we choose a generator of $\mathfrak{g}_2$, namely X_ψ , and we let for $X \in \mathfrak{g}_{-1}$,

$$P(X) = K(X_\psi, ad(X)^4 X_\psi).$$

This is obviously a G_0 -invariant polynomial, hence the equation of an invariant hypersurface, which must be Y^* .

The open orbit in $\mathbb{P}\mathfrak{g}_{-1}$ is the complement of this quartic hypersurface, and the associated birational map is a cubo-cubic Cremona transformation of $\mathbb{P}\mathfrak{g}_1$.

Starting from the exceptional Lie algebras, we get five exceptional Legendrian varieties that we can relate to the Jordan algebras $\mathcal{J}_3(\mathbb{A})$. Observe that $\mathfrak{g}_{-1}$ has dimension $6a + 8$, which is twice the dimension of $\mathcal{J}_3(\mathbb{A})$ plus two. In fact one can make an identification

$$\mathfrak{g}_{-1} \simeq \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathbb{C}$$

in such a way that the following statement does hold:

Proposition 5.3.6. *The Legendrian varieties associated to the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are the twisted cubic “curves” over the Jordan algebras $\mathcal{J}_3(\mathbb{A})$, for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.*

By twisted cubic “curves” over the Jordan algebras $\mathcal{J}_3(\mathbb{A})$ we mean the following. Recall that we have defined a “determinant” on $\mathcal{J}_3(\mathbb{A})$, whose existence we related to the Cayley-Sylvester identity $X^3 - f_1(X)X^2 + f_2(X)X - f_3(X)I = 0$, for $X \in \mathcal{J}_3(\mathbb{A})$. We have $f_1(X) = \text{Trace}(X)$ and $f_3(X) = \text{Det}(X)$ is our determinant. As usual we can rewrite the previous identity as $X \text{Com}(X) = \text{Com}(X)X = f_3(X)I$ where the “comatrix” $\text{Com}(X) = X^2 - f_1(X)X + f_2(X)I$ is to be thought of as giving the “inverse” matrix when X is “invertible”. Then the map

$$X \in \mathcal{J}_3(\mathbb{A}) \mapsto [1, X, \text{Com}(X), \text{Det}(X)] \in \mathbb{P}\mathfrak{g}_{-1}$$

parametrizes a dense open subset of the Legendrian variety $Y = Y(\mathbb{A})$. This description extends to $\mathfrak{so}_8$ if we let $\mathbb{A} = 0$, and also to $\mathfrak{g}_2$ for $\mathbb{A} = \Delta$; in the latter case the associated Legendrian variety is the usual twisted cubic in $\mathbb{P}^3$!

A more explicit description of the varieties $Y(\mathbb{A})$ is as follows:

$$\begin{array}{ll} Y(\Delta) &= v_3(\mathbb{P}^1), & Y(\mathbb{C}) &= G(3, 6), \\ Y(0) &= \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, & Y(\mathbb{H}) &= OG(6, 12), \\ Y(\mathbb{R}) &= LG(3, 6), & Y(\mathbb{O}) &= E_7/P_7. \end{array}$$

Using (for example) the description above one can show that:

Proposition 5.3.7. *The Legendrian variety Y has the following properties:*

- (1) *It is non degenerate, in the sense that its secant variety is the whole projective space $\mathbb{P}\mathfrak{g}_{-1}$.*
- (2) *The tangent variety of Y coincides with its dual hypersurface.*
- (3) *Any point outside the tangent variety belongs to a unique secant line to Y ; in particular Y is a variety with one apparent double point.*

- (4) *Any point on the smooth locus of the tangent variety belongs to a unique tangent to Y .*

One can wonder if these properties can be used to approach the LeBrun-Salamon conjecture. Starting from a Fano contact manifold X with its contact distribution H , one can look at rational curves of minimal degree through a general point x of X . Their tangent directions define a subvariety $Y_x \subset \mathbb{P}H_x \subset \mathbb{P}T_x X$. Kebekus proved that Y_x is a smooth Legendrian variety. It would be nice to prove that Y_x is one of the homogeneous Legendrian varieties, and then to deduce that X itself must be homogeneous. A step in this direction is the following statement.

Proposition 5.3.8 (Buczynski). *Let $Y \subset \mathbb{P}V$ be a Legendrian variety whose ideal is generated by quadrics. Then Y must be homogeneous.*

5.3.3. Severi varieties. Starting from the homogeneous Legendrian varieties we can apply once more our reduction procedure. In the exceptional cases, let us consider our description of these varieties as twisted cubics over the Jordan algebras $\mathcal{J}_3(\mathbb{A})$. It is clear from this description that the tangent space at the point $[1, 0, 0, 0]$ is precisely $\mathcal{J}_3(\mathbb{A})$. So by reduction we get a subvariety $Z(\mathbb{A}) \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$, with an action of a Lie group $H(\mathbb{A})$.

Proposition 5.3.9. *The Lie group $H(\mathbb{A})$ is the subgroup of $GL(\mathcal{J}_3(\mathbb{A}))$ preserving the determinant. Its orbit closures in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ are*

$$Z(\mathbb{A}) \subset \text{Sec}(Z(\mathbb{A})) \subset \mathbb{P}\mathcal{J}_3(\mathbb{A}),$$

to be considered as the sets of matrices of "rank" at most one, two and three respectively.

In particular $Z(\mathbb{A})$ is smooth and is acted on transitively by $H(\mathbb{A})$. Its equations are the " 2×2 minors", the derivatives of the determinant. In particular one can show that a dense open subset of $Z(\mathbb{A})$ can be parametrized as the set of matrices of the form

$$\begin{pmatrix} 1 & u & v \\ \bar{u} & |u|^2 & \bar{u}v \\ \bar{v} & \bar{v}u & |v|^2 \end{pmatrix}, \quad u, v \in \mathbb{A}.$$

This implies that $Z(\mathbb{A})$ has dimension $n = 2a$, but $\text{Sec}(Z(\mathbb{A}))$ has dimension $3a+1 = \frac{3}{2}n + 1$, while we would expect that $\text{Sec}(Z(\mathbb{A})) = \mathbb{P}\mathcal{J}_3(\mathbb{A})$.

Theorem 5.3.10 (Zak). *Let $Z \subset \mathbb{P}^N$ be a smooth, linearly non degenerate variety, with $N < \frac{3}{2}n + 2$. Then $\text{Sec}(Z) = \mathbb{P}^N$.*

Moreover, if $N < \frac{3}{2}n + 2$ and $\text{Sec}(Z) \neq \mathbb{P}^N$, then Z is one of the $Z(\mathbb{A})$.

Explicitly, we get the four *Severi varieties*

$$Z(\mathbb{R}) = v_2(\mathbb{P}^2), \quad Z(\mathbb{C}) = \mathbb{P}^2 \times \mathbb{P}^2, \quad Z(\mathbb{H}) = G(2, 6), \quad Z(\mathbb{O}) = E_6/P_1.$$

Of course the secant variety $\text{Sec}(Z(\mathbb{A}))$ is the cubic hypersurface defined by the determinant. The derivatives of this polynomial define a quadro-quadric Cremona transformation of $\mathbb{P}\mathcal{J}_3(\mathbb{A}) = \mathbb{P}^{3a+2}$, whose base locus is exactly $Z(\mathbb{A})$. In particular this base locus is smooth and connected. This is again a very exceptional property.

Theorem 5.3.11 (Ein & Shepherd-Barron). *Consider a quadro-quadric Cremona transformation, and suppose that the base locus is smooth and connected. Then it must be defined by the quadrics containing one of the Severi varieties.*

5.3.4. Conclusion. We have sketched a kind of geometric version of the Tits-Freudenthal magic square by associating to any pair of normed algebras a variety $X(\mathbb{A}, \mathbb{B})$ whose automorphism group has Lie algebra $\mathfrak{g}(\mathbb{A}, \mathbb{B})$.

	Δ	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
sub-Severi	$\emptyset$	$\emptyset$	$v_4(\mathbb{P}^1)$	$F_{1,2}$	$IG(2, 6)$	F_4/P_1
Severi	$\emptyset$	3 pts	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2, 6)$	E_6/P_1
Legendrian	$v_3(\mathbb{P}^1)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$LG(3, 6)$	$G(3, 6)$	$OG(6, 12)$	E_7/P_7
adjoint	$X_{ad}(\mathfrak{g}_2)$	$X_{ad}(\mathfrak{so}_8)$	$X_{ad}(\mathfrak{f}_4)$	$X_{ad}(\mathfrak{e}_6)$	$X_{ad}(\mathfrak{e}_7)$	$X_{ad}(\mathfrak{e}_8)$

The projective varieties in this table have extremely special and particularly interesting properties. Moreover:

- (1) Their geometric properties are essentially the same in a given line.
- (2) One can pass from one line to another by the reduction procedure, which is the geometric version of the algebraic reduction that passes from a $\mathbb{Z}$ -graded Lie algebra to the degree one component of this grading.

In fact one can associate to each box of the magic square not only one but several projective varieties with astonishing relations between them.

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